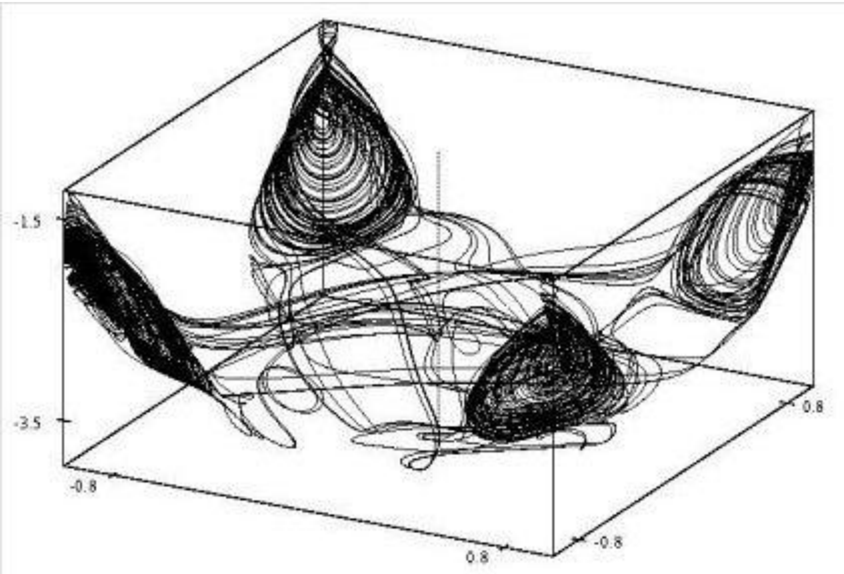
Lyapunov exponents



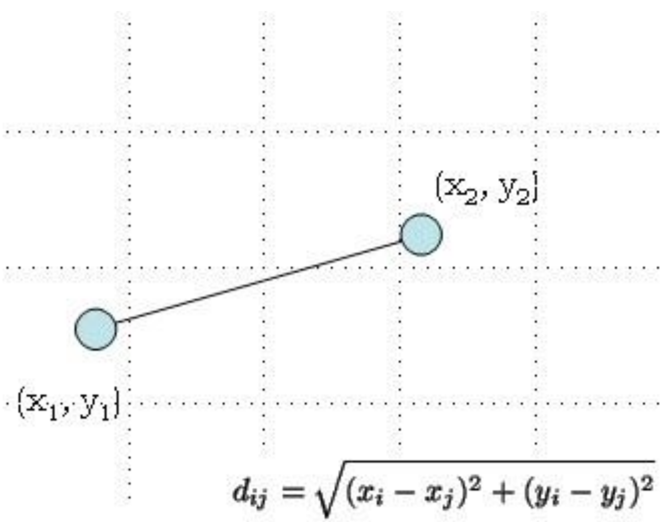
Lyapunov exponents quantify the average exponential separation between nearby phase space trajectories.

A dynamical System in $\boldsymbol{R}^m$ has m Lyapunov exponents.

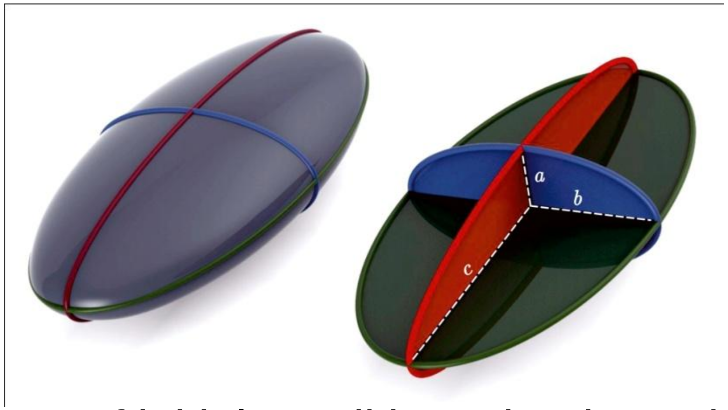
The maximum Lyapunov exponent $\lambda(x_0)$ with respect to a reference orbit x_0 determines the system behaviour and is given by:

$$\lambda(\vec{x}_0) = \lim_{t\to\infty} \lim_{\|\Delta \vec{x}(0)\|\to 0} \frac{1}{t} \log \frac{\|\Delta \vec{x}(t)\|}{\|\Delta \vec{x}(0)\|}$$

$\|\Delta \vec{x}(0)\|$ is the Euclidian distance between the trajectories $x_0(t)$ and $x_1(t)$ at initial time t=0



Lyapunov exponents



Consider an infinitesimal m-dimensional sphere of initial conditions that is anchored to a reference trajectory. As the trajectory evolves in time, it becomes deformed into an ellipsoid.

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \log\left(\frac{p_i(t)}{p_i(0)}\right) \qquad i = 1, 2, \ldots, m$$

with p_i is the length of the i-th principal axis

Volume elements in phase space evolve in time as

$$V(t) \sim V(0) \exp\left(\sum_i^m \lambda_i t\right)$$

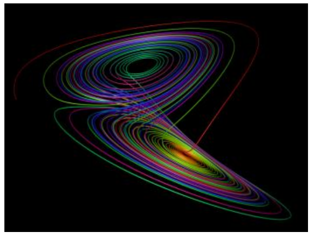
The sum of the Lyapunov exponents is equal to the divergence of the vector field

$$\sum_i^m \lambda_i = \vec{\nabla} . \vec{F}$$

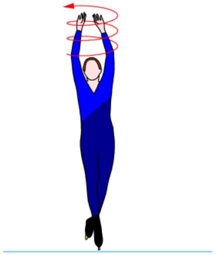
Lyapunov exponents

If the largest Lyapunov exponent is positive, trajectories will diverge:
→ Chaotic system

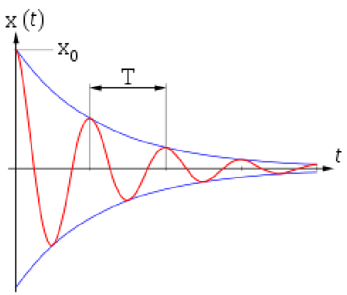




Otherwise, they will get closer reaching a non chaotic attractor. We have either a conservative system ($\lambda = 0$) in which no work is done on a closed trajectory (gravity, conservation of angular momentum...)




or a dissipative system for ($\lambda < 0$) in which energy (internal, bulk flow kinetic, or system potential) is transformed from an initial form to a final form (dampened oscillation)



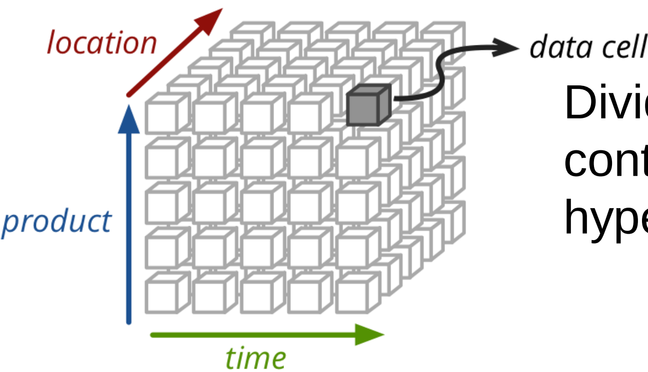
Lyapunov exponents

Following this argument, a necessary condition for a system to be chaotic is that at least one of the exponents (the largest one) is positive.

Lyapunov exponents also give an indication of the period of time in which predictions are possible and this is strongly related with the concept of information theory and entropy:

The sum of all positive Lyapunov exponents (expansion rate of the manifold) equals the Kolmogorov entropy

$$K_2 = \sum_{\lambda>0} \lambda_i$$



Divide phase space into D-dimensional hypercubes of ε^D content . Let $P_{i0\ldots in}$ be the probability that a trajectory is in hypercube i_0 at t=0, i_1 at t=T, i_2 at t=2T , etc.

$$K_n = h_K = -\sum_{i_0,\ldots,i_n} P_{i_0,\ldots,i_n} \ln P_{i_0,\ldots,i_n},$$

$$K \equiv \lim_{T\to 0} \lim_{\epsilon\to 0^+} \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} (K_{n+1} - K_n).$$

Correlation Dimension D_2

In chaos theory, the **correlation dimension D_2** is a measure of the dimensionality of the space occupied by a set of random points, often referred to as a type of fractal dimension → attractors

A set of points distributed on a triangle, embedded in a cubic space: $D_2 = 2$

Estimation by Grassberger-Procaccia Algorithm

The probability that two points of the set are in the same cell of size r is approximately equal to the probability that two points of the set are separated by a distance ρ less than or equal to r:

$$C(r) \approx \frac{\sum_{i=1,j>i}^{N} \Theta\left(r - \rho(\vec{x}_i, \vec{x}_j)\right)}{\frac{1}{2}N(N-1)}$$

with Θ being the Heaviside function

$$\Theta(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$$

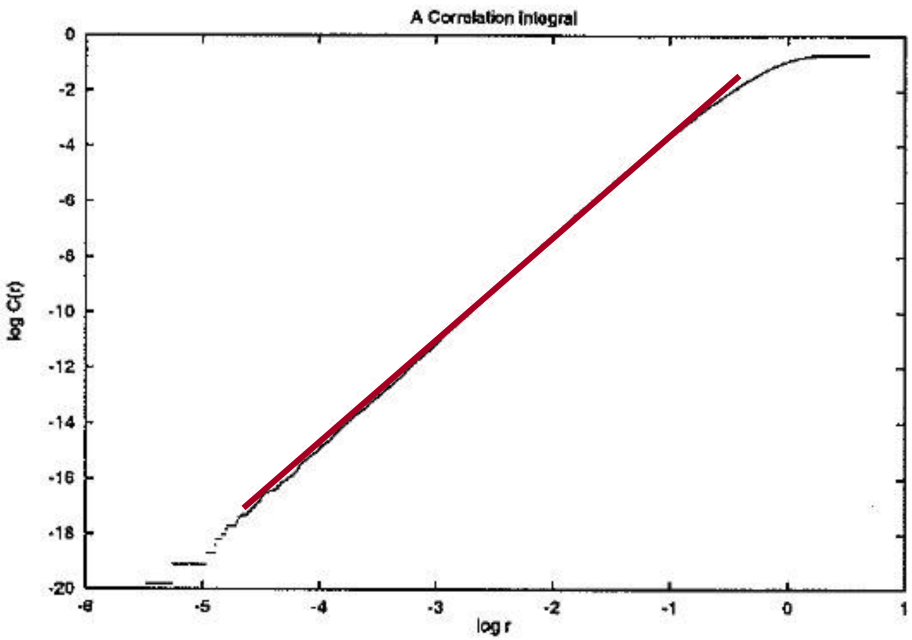
Euclidean distance

$$\rho(\vec{x}_i, \vec{x}_j) = \sqrt{\sum_{k=1}^{m} \left(x_i(k) - x_j(k)\right)^2}$$

Correlation Dimension D_2

The approximation made is exact in the limit $N \rightarrow \infty$; however, this limit cannot be realized in practical applications. The limit $r \rightarrow 0$ used in the definition of D_2 is also not possible in practice.

Instead, Procaccia and Grassberger propose the (approximate) evaluation of C(r) over a range of values of r and then deduce D_2 from the slope of the straight line of best fit in the linear scaling region of a plot of log C(r) versus log r.



Practical examples:

Epilepsy:

Petit Mal (Babloyantz and Destexhe, 1986): Druing seizures attractor has a global stability (low D2) but $\lambda = 2.9 \pm 0.6$ → chaotic properties and great sensitivity to initial conditions

Grand Mal (Iasemidis and Sackellares, 1991): Drop in the Lyapunov exponents during seizures but higher values postictally (chaotic state) than ictally or pre-ictally.

Sleep:

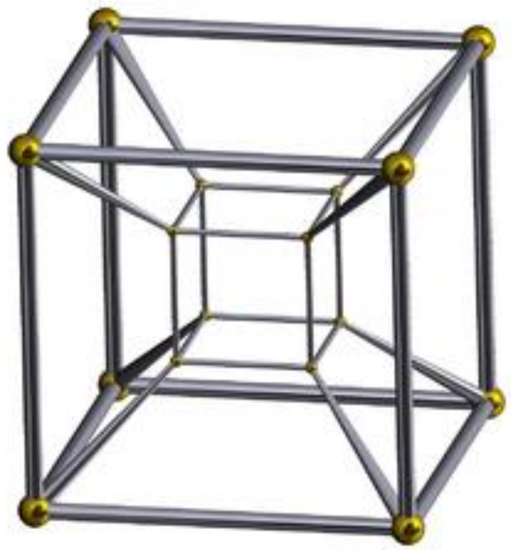
Babloyantz 1988, Röschke 1994: Lyapunov exponents positive but decrease as sleep becomes slower (Stage II: $\lambda = 0.6$ +- 0.2, Stage IV: $\lambda = 0.45$ +- 0.15)

Dementia and Parkinson:

Stam, 1995: Compared 13 Parkinson and 9 demented patients against a healthy control group. They found $\lambda = 6.17$ for heathy and $\lambda = 6.12$ for Parkinson (but lower D2) and a significant lower $\lambda = 4.84$ for demented patients.

Limitations:

Lyapunov exponents sensible to evolution time[1] and embedding dimension[2]



1: If time steps are chosen too small no evolution of neighbor trajectories (e.g., sticky orbits), if chosen too large jumps to other trajectories give unreliable results.

2: Need for a complete unfolding of the attractor → testing of multiple embedding dimensions