Computational Diagnostics & Biocybernetics Unit

BMT 801 : Biomedizinische Signal- und Bildverarbeitung EF2

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The following slides were partly prepared by R. Polikar, Rutgers University, New Brunswick, NJ

The Story of Wavelets

- **Technical Overview**
  - But…We cannot do that with Fourier Transform….
  - Time - frequency representation and the STFT
  - Continuous wavelet transform
  - Multiresolution analysis and discrete wavelet transform (DWT)
- **Application Overview**
  - Conventional Applications: Data compression, denoising, solution of PDEs, biomedical signal analysis.
  - Unconventional applications
  - Yes…We can do that with wavelets too…
- **Historical Overview**
  - 1807 ~ 1940s: The reign of the Fourier Transform
  - 1940s ~ 1970s: STFT and Subband Coding
  - 1980s & 1990s: The Wavelet Transform and MRA

What is a Transform and Why Do we Need One ?

- **Transform:** A mathematical operation that takes a function or sequence and maps it into another one
- Transforms are good things because…
  - The transform of a function may give additional /hidden information about the original function, which may not be available /obvious otherwise
  - The transform of an equation may be easier to solve than the original equation (recall your fond memories of Laplace transforms in DFQs)
  - The transform of a function/sequence may require less storage, hence provide data compression / reduction
  - An operation may be easier to apply on the transformed function, rather than the original function (recall other fond memories on convolution).

December, 21, 1807

Jean B. Joseph Fourier
(1768-1830)

"An arbitrary function, continuous or with discontinuities, defined in a finite interval by an arbitrarily capricious graph can always be expressed as a sum of sinusoids"

J.B.J. Fourier

$$\textit{Complex Function} = \sum_i (\textit{weight})_i \bullet (\textit{Simple Function})_i$$

- Complex function representation through simple building blocks
  - Basis functions
- Using only a few blocks ➔ Compressed representation
- Using sinusoids as building blocks ➔ Fourier transform
  - Frequency domain representation of the function

$$F(\omega) = \int f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi} \int F(\omega) e^{j\omega t} d\omega$$

How Does FT Work Anyway?

- Recall that FT uses complex exponentials (sinusoids) as building blocks.

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- For each frequency of complex exponential, the sinusoid at that frequency is compared to the signal.

- If the signal consists of that frequency, the correlation is high → large FT coefficients.

$$F(\omega) = \int f(t)e^{-j\omega t}dt \qquad f(t) = \frac{1}{2\pi}\int F(\omega)e^{j\omega t}d\omega$$

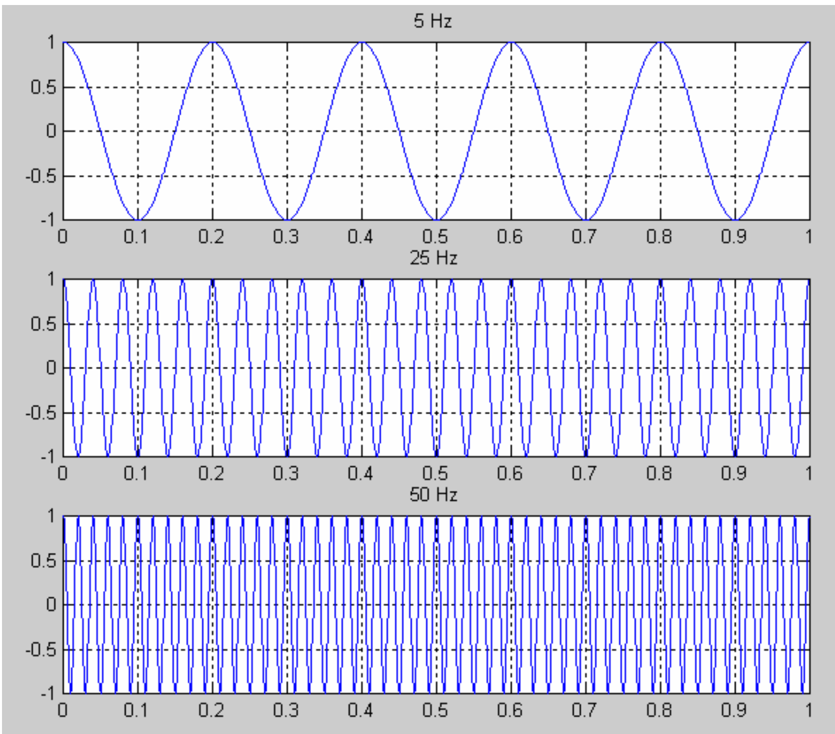
- If the signal does not have any spectral component at a frequency, the correlation at that frequency is low / zero, → small / zero FT coefficient.

FT At Work

$$x_1(t) = \cos(2\pi \cdot 5 \cdot t)$$

$$x_2(t) = \cos(2\pi \cdot 25 \cdot t)$$

$$x_3(t) = \cos(2\pi \cdot 50 \cdot t)$$

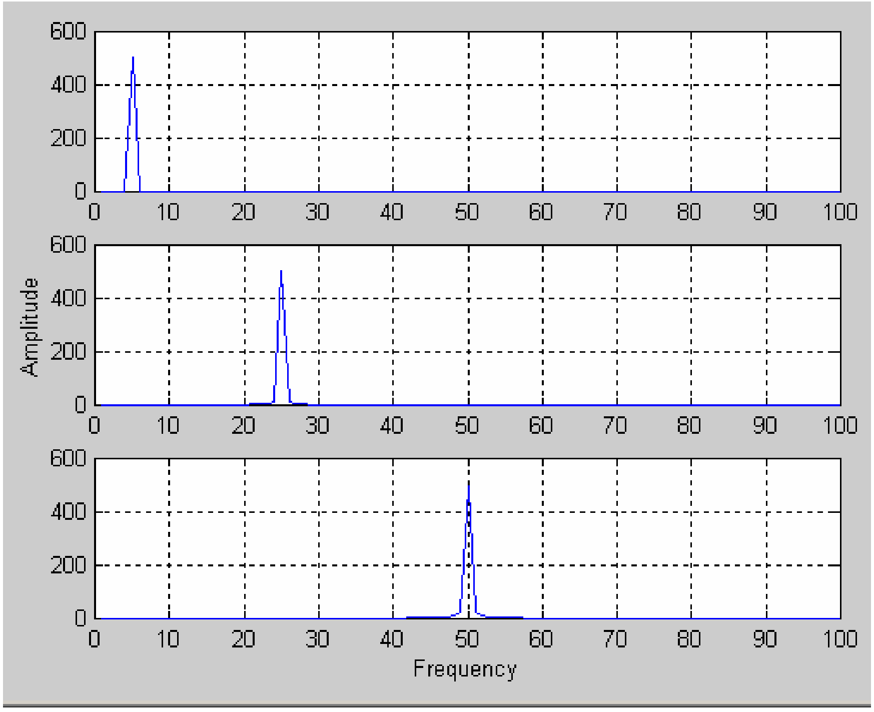


FT At Work

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)$$

$$x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

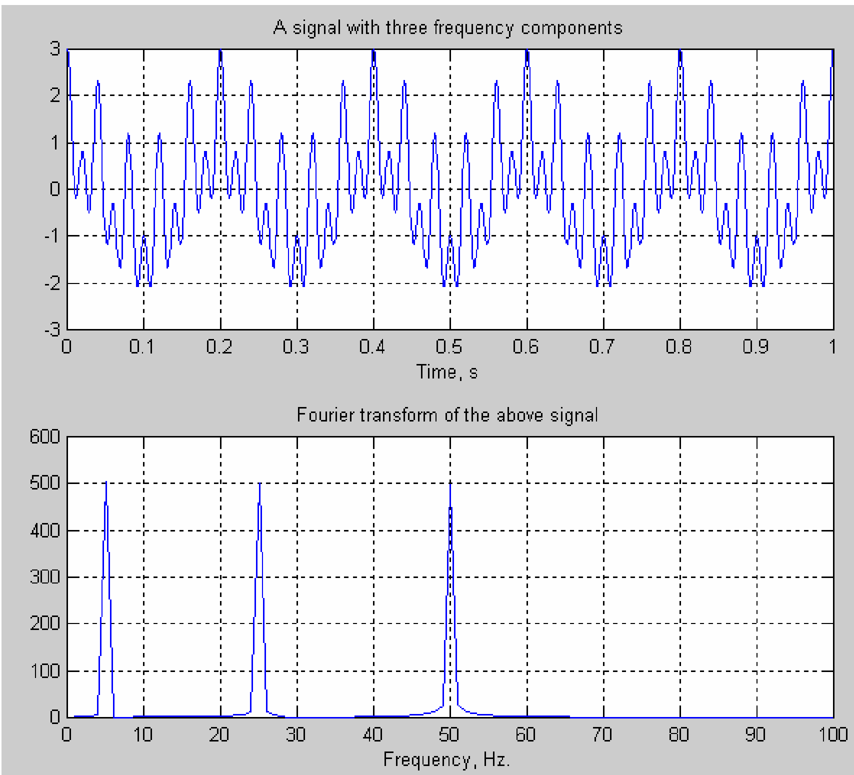
$$x_3(t) \overset{\mathcal{F}}{\longleftrightarrow} X_3(\omega)$$



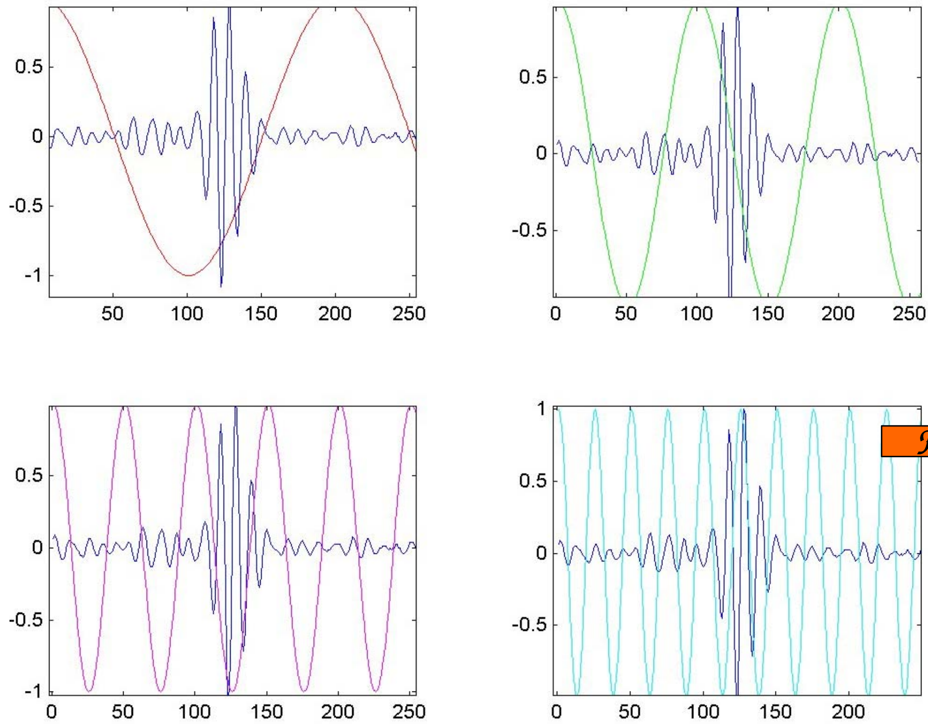
FT At Work

$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$

$$x_4(t) \overset{\mathcal{F}}{\longleftrightarrow} X_4(\omega)$$



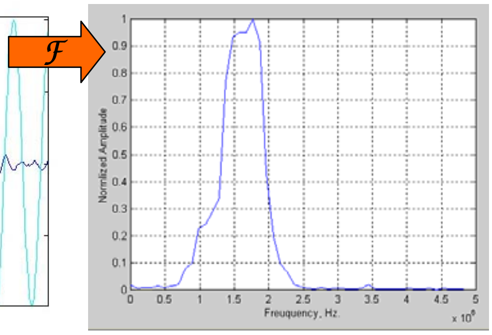
FT At Work



Complex exponentials (sinusoids) as basis functions:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} dt$$



An ultrasonic A-scan using 1.5 MHz transducer, sampled at 10 MHz

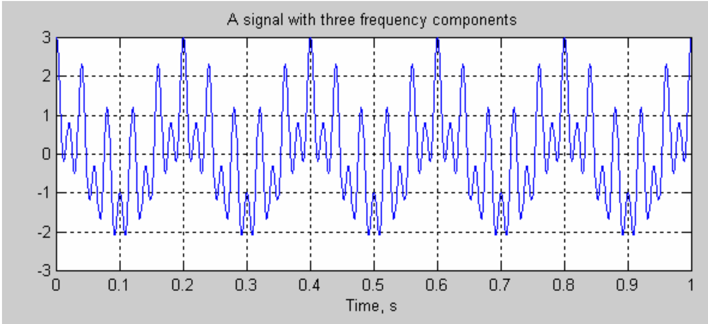
Stationary and Non-stationary Signals

- FT identifies all spectral components present in the signal, however it does not provide any information regarding the temporal (time) localization of these components. Why?
- Stationary signals consist of spectral components that do not change in time
  - all spectral components exist at all times
  - no need to know any time information
  - FT works well for stationary signals
- However, non-stationary signals consists of time varying spectral components
  - How do we find out which spectral component appears when?
  - FT only provides ***what spectral components exist*** , not where in time they are located.
  - Need some other ways to determine ***time localization of spectral components***

Stationary and Non-stationary Signals

- Stationary signals' spectral characteristics do not change with time

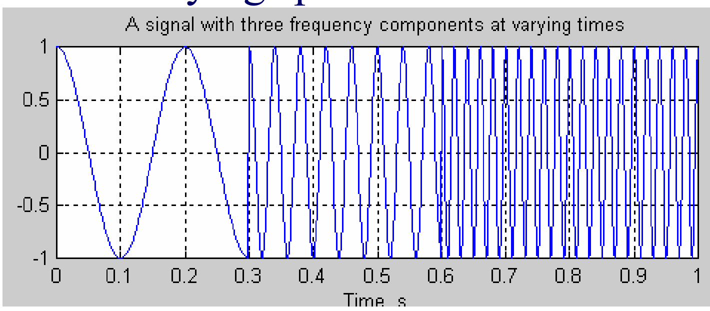
$$x_4(t) = \cos(2\pi \cdot 5 \cdot t) + \cos(2\pi \cdot 25 \cdot t) + \cos(2\pi \cdot 50 \cdot t)$$



- Non-stationary signals have time varying spectra

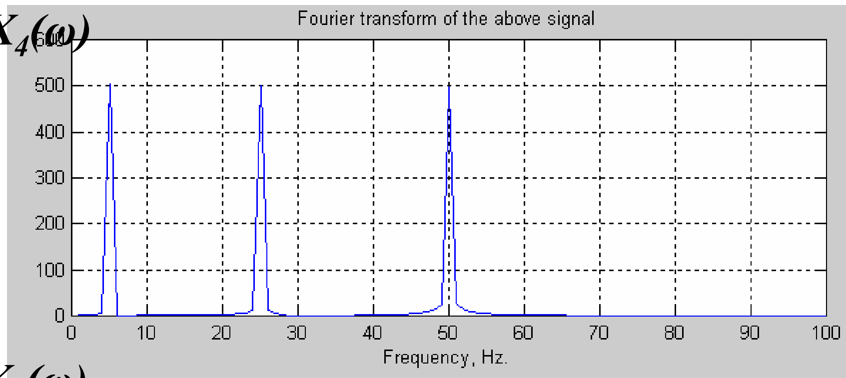
$$x_5(t) = [x_1 \oplus x_2 \oplus x_3]$$

$\oplus$ Concatenation



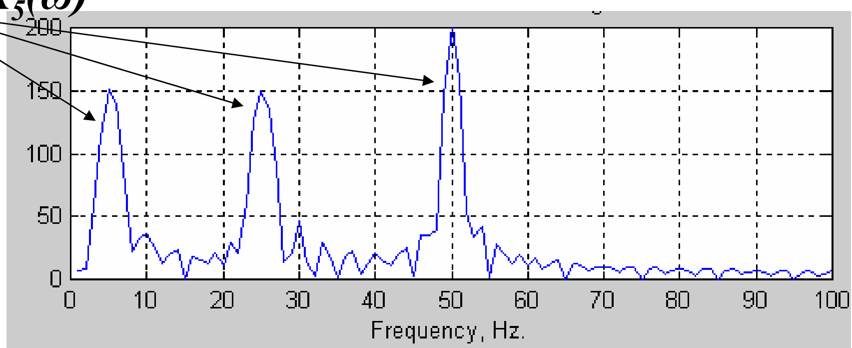
Stationary vs. Non-Stationary

$X_4(\omega)$



$X_5(\omega)$

Perfect knowledge of what frequencies exist, but no information about where these frequencies are located in time



SHORTCOMINGS OF THE FT

- Sinusoids and exponentials
  - Stretch into infinity in time, ➡ no time localization
  - Instantaneous in frequency, ➡ perfect spectral localization
  - ***Global*** analysis does not allow analysis of non-stationary signals
- Need a ***local*** analysis scheme for a time-frequency representation (TFR) of nonstationary signals
  - Windowed F.T. or Short Time F.T. (STFT) : Segmenting the signal into narrow time intervals, narrow enough to be considered stationary, and then take the Fourier transform of each segment, Gabor 1946.
  - Followed by other TFRs, which differed from each other by the selection of the windowing function

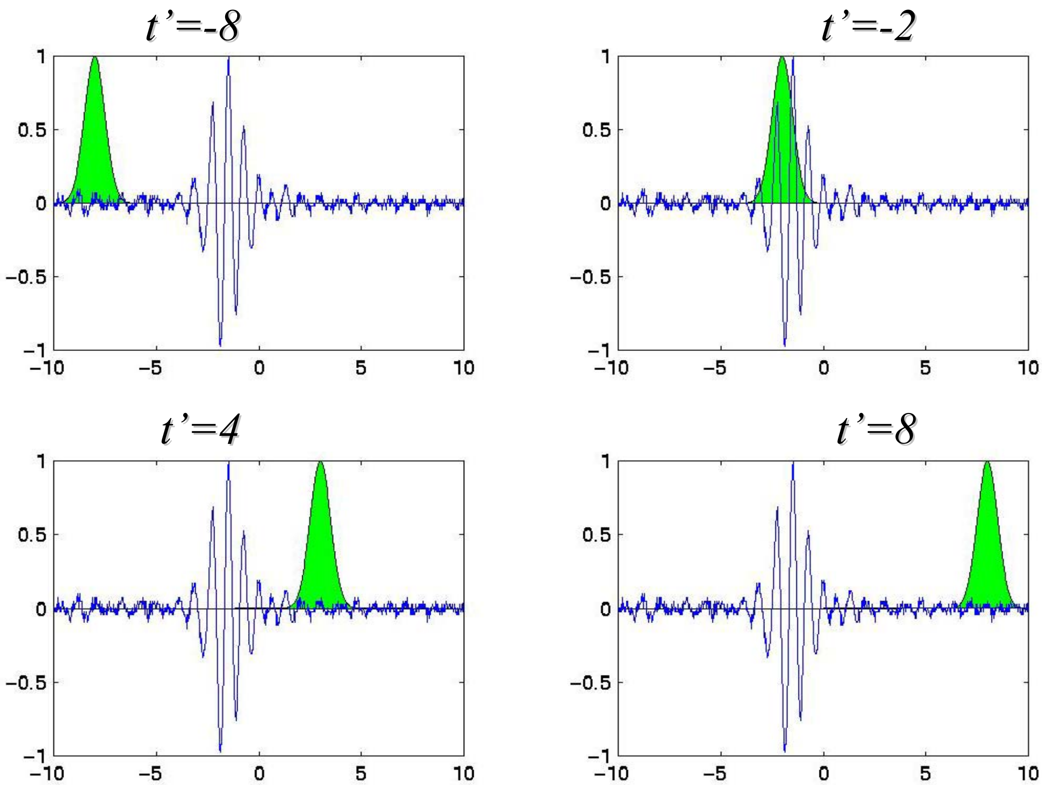
Short Time Fourier Transform (STFT)

1. Choose a window function of finite length
2. Place the window on top of the signal at t=0
3. Truncate the signal using this window
4. Compute the FT of the truncated signal, save.
5. Incrementally slide the window to the right
6. Go to step 3, until window reaches the end of the signal

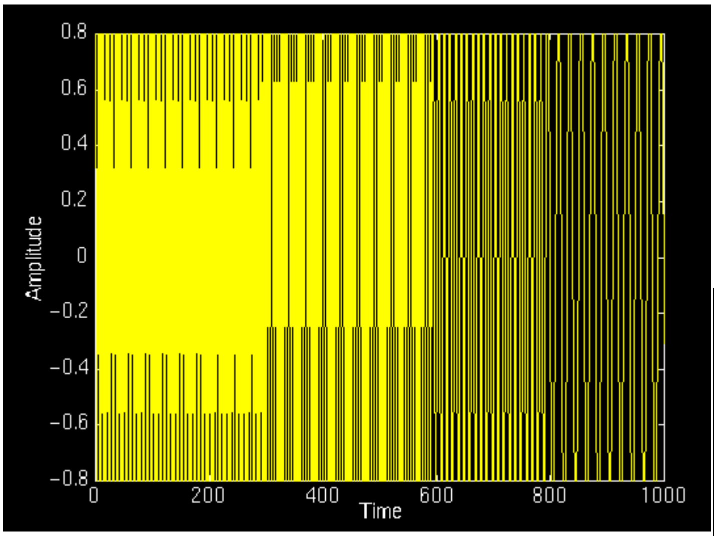
➲ For each time location where the window is centered, we obtain a different FT

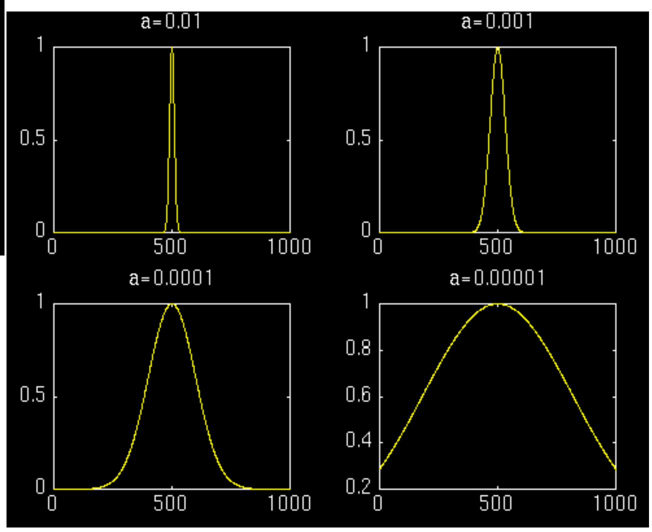
- Hence, each FT provides the spectral information of a separate time-slice of the signal, providing simultaneous time and frequency information

STFT

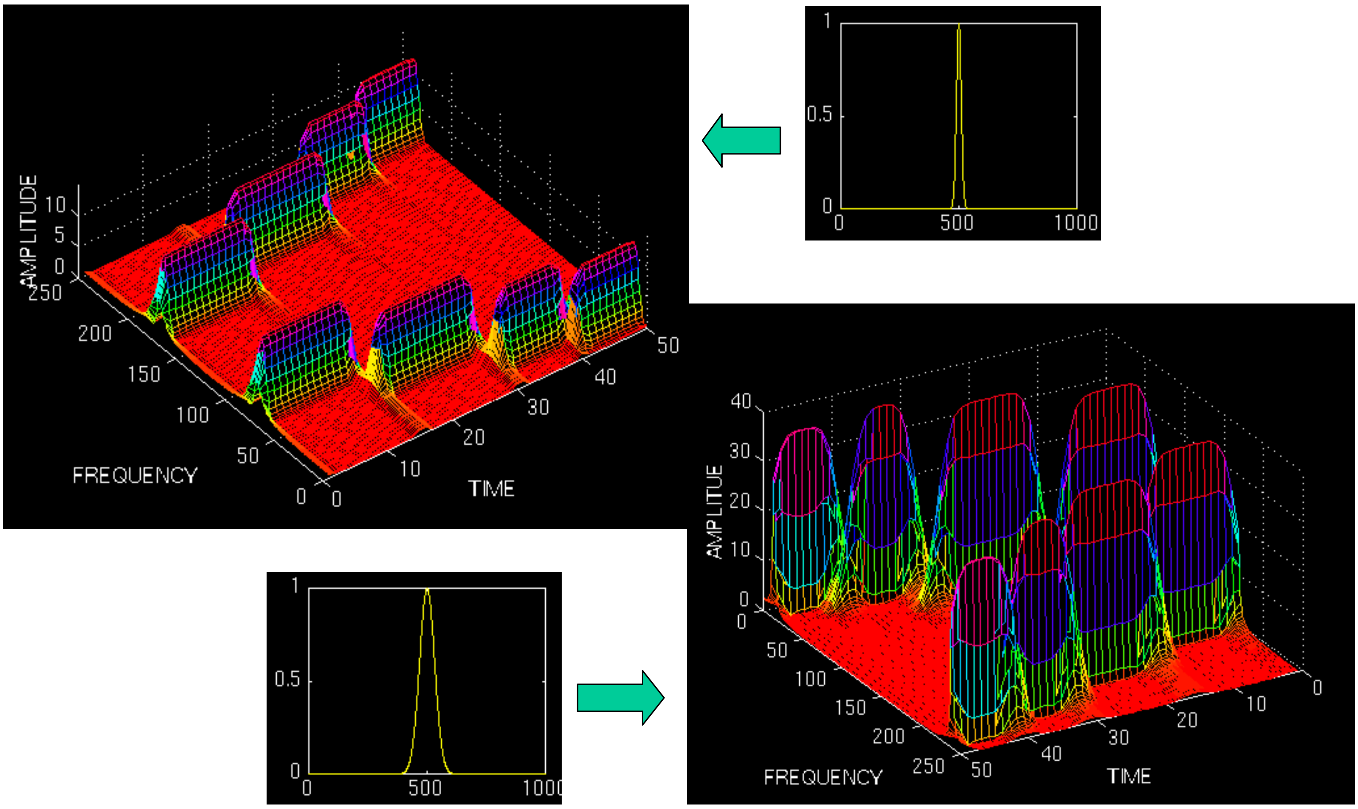


STFT at Work

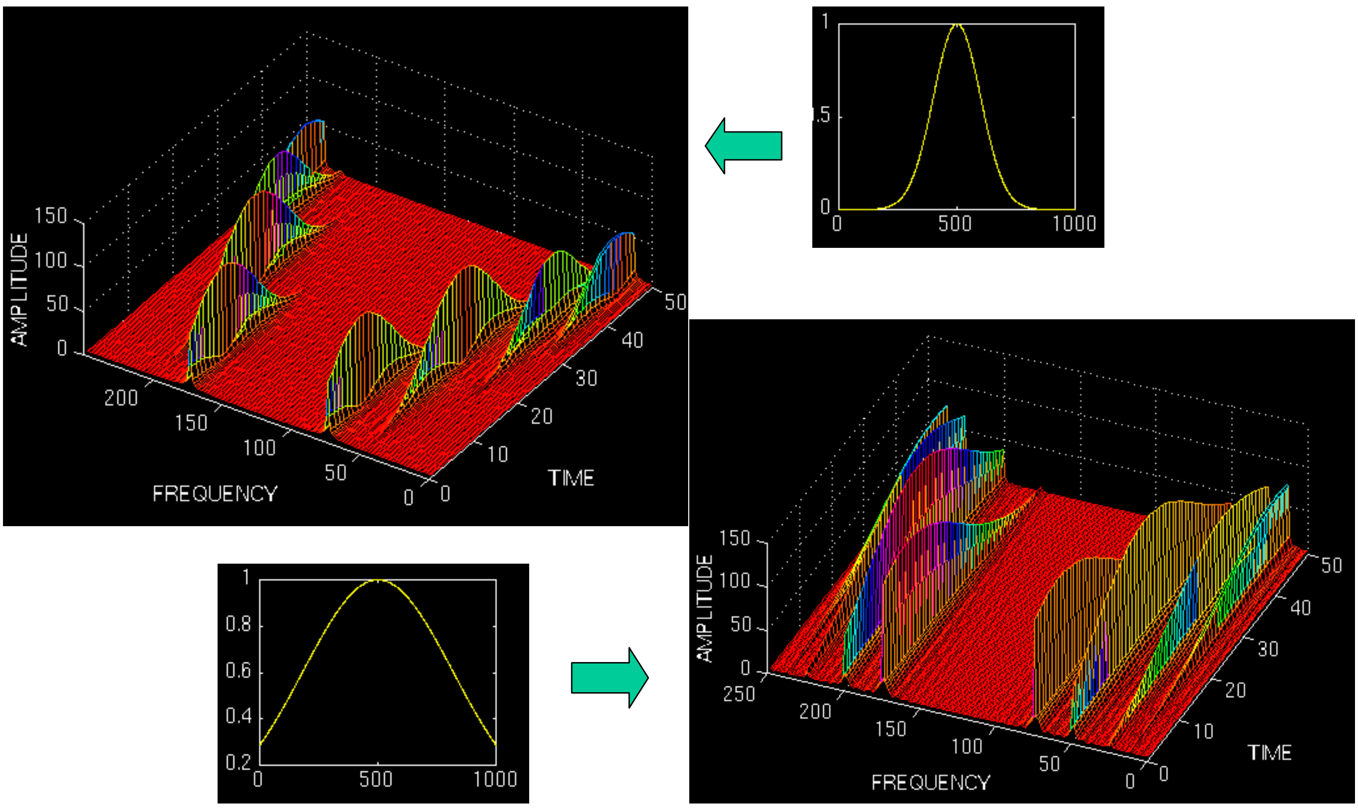




STFT At Work



STFT At Work



STFT

- STFT provides the time information by computing a different FTs for consecutive time intervals, and then putting them together
  - Time-Frequency Representation (TFR)
  - Maps 1-D time domain signals to 2-D time-frequency signals
- Consecutive time intervals of the signal are obtained by truncating the signal using a sliding windowing function
- How to choose the windowing function?
  - What shape? Rectangular, Gaussian, Elliptic…?
  - How wide?
    - Wider window require less time steps → low time resolution
    - Also, window should be narrow enough to make sure that the portion of the signal falling within the window is stationary
    - Can we choose an arbitrarily narrow window…?

Selection of STFT Window

Two extreme cases:

- *W(t)* infinitely long: $W(t) = 1$ → STFT turns into FT, providing excellent frequency information (good frequency resolution), but no time information
- *W(t)* infinitely short: $W(t) = \delta(t)$

$$STFT_x^{\omega}(t', \omega) = \int_t \left[x(t) \cdot \delta(t - t')\right] \cdot e^{-j\omega t} dt = x(t') \cdot e^{-j\omega t'}$$

→ STFT then gives the time signal back, with a phase factor. Excellent time information (good time resolution), but no frequency information

Wide analysis window → poor time resolution, good frequency resolution

Narrow analysis window → good time resolution, poor frequency resolution

Once the window is chosen, the resolution is set for both time and frequency.

Heisenberg Principle

$$\Delta t \cdot \Delta f \geq \frac{1}{4\pi}$$

Time resolution: How well two spikes in time can be separated from each other in the transform domain

Frequency resolution: How well two spectral components can be separated from each other in the transform domain

Both time and frequency resolutions cannot be arbitrarily high!!!

➔ ➔We cannot precisely know at what time instance a frequency component is located. We can only know what *interval of frequencies* are present in which *time intervals*

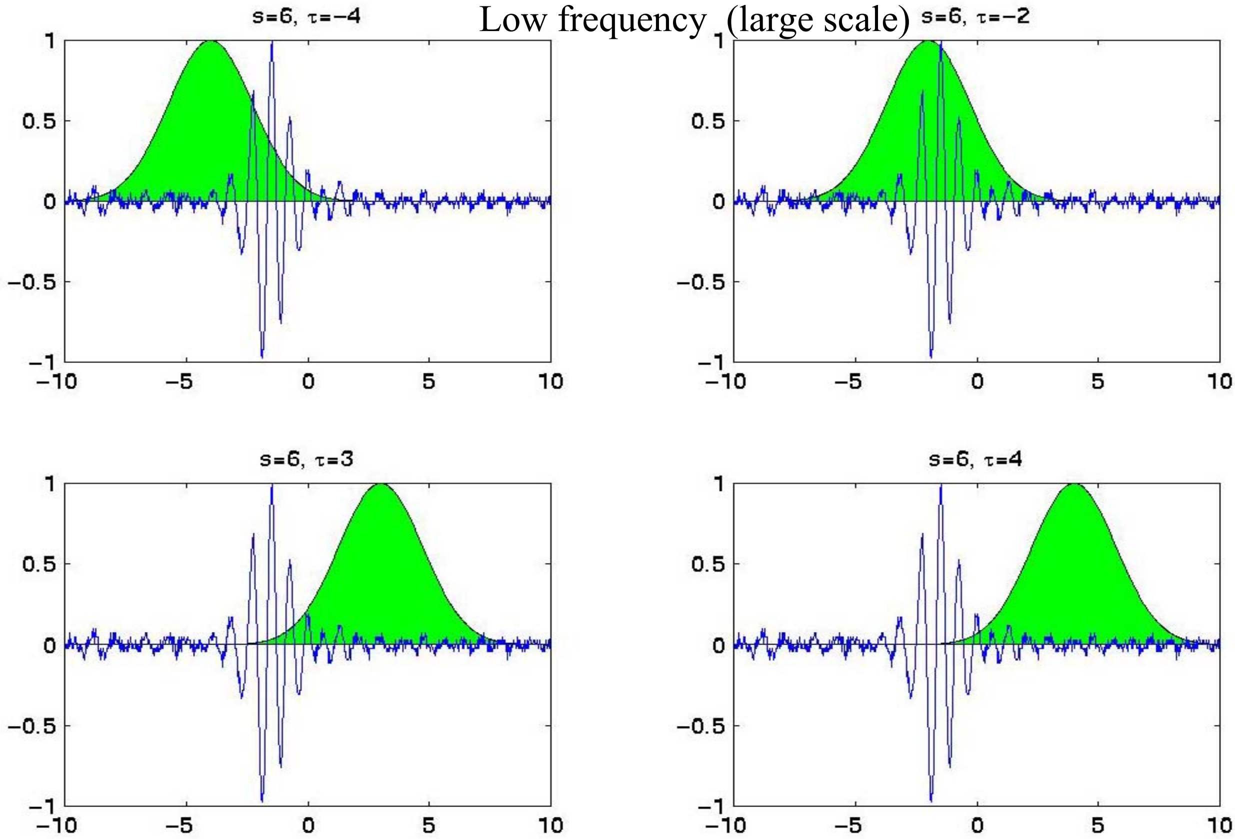
The Wavelet Transform

- Overcomes the preset resolution problem of the STFT by using a variable length window
- Analysis windows of different lengths are used for different frequencies:
  - Analysis of high frequencies➔ Use narrower windows for better time resolution
  - Analysis of low frequencies ➔ Use wider windows for better frequency resolution
- This works well, if the signal to be analyzed mainly consists of slowly varying characteristics with occasional short high frequency bursts.
- Heisenberg principle still holds!!!
- The function used to window the signal is called ***the wavelet***

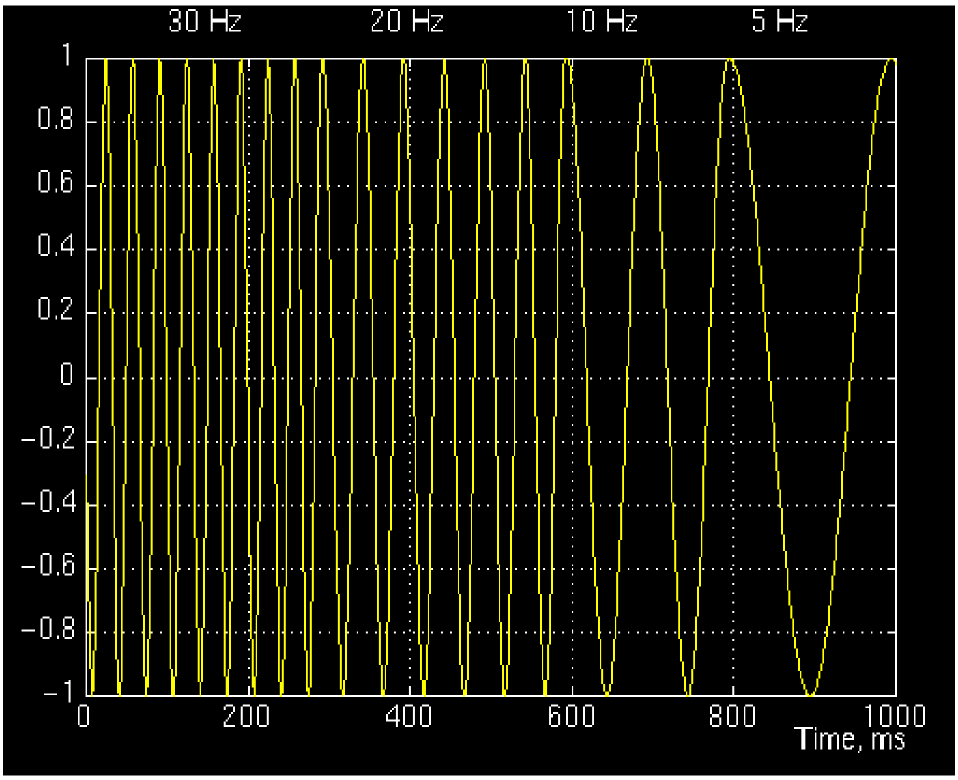
See Blackboard for the Mathematics

WT at Work

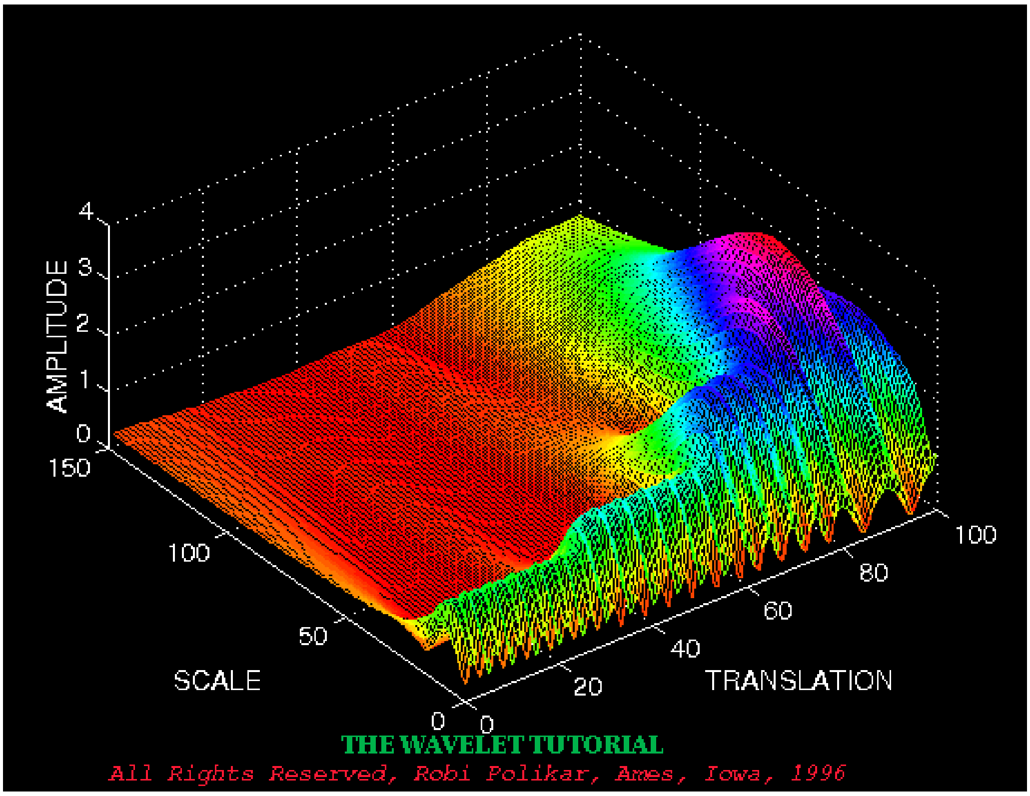
Low frequency (large scale)



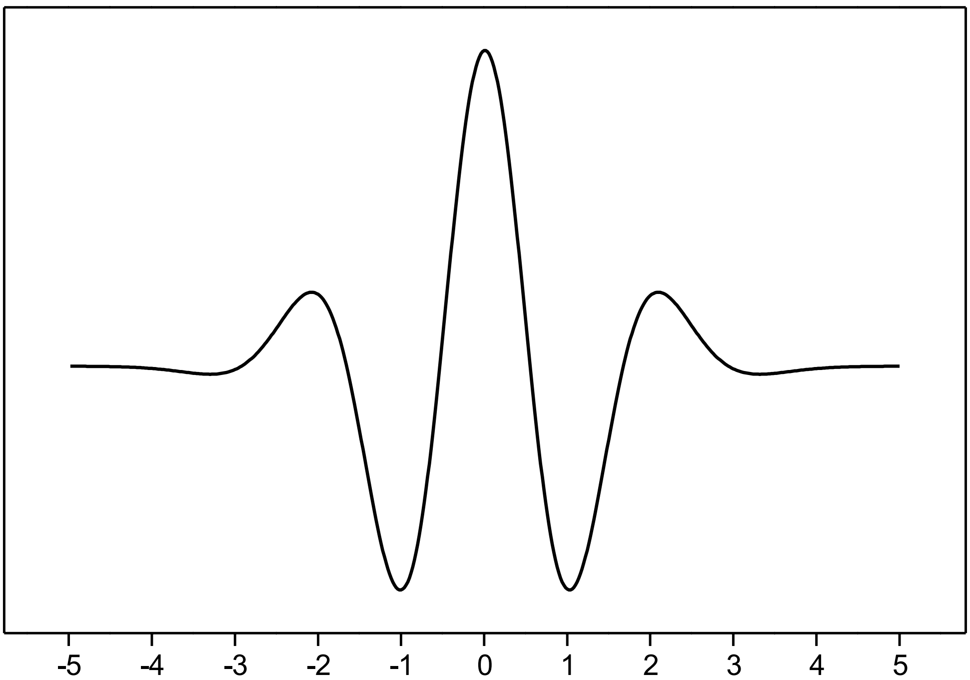
WT at Work



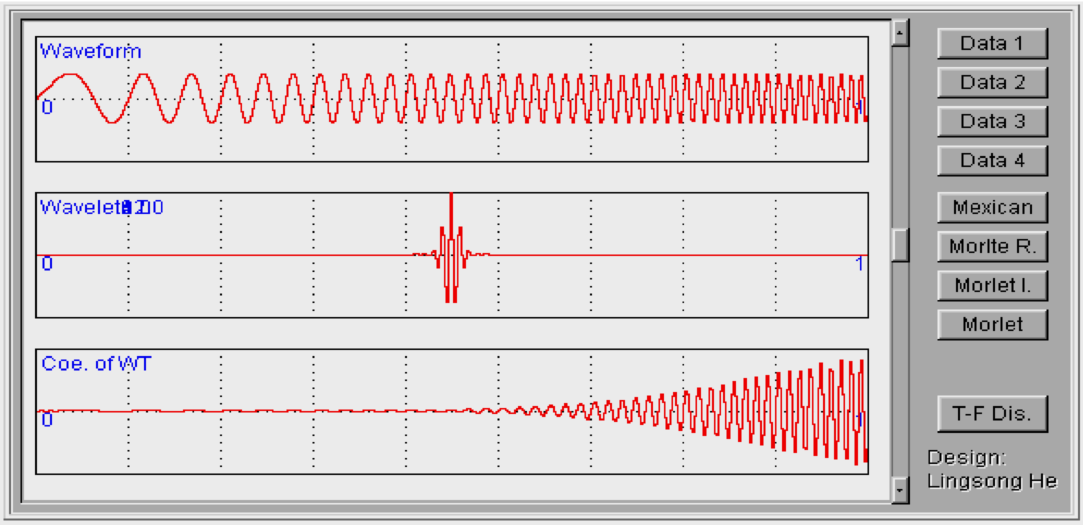
WT at Work

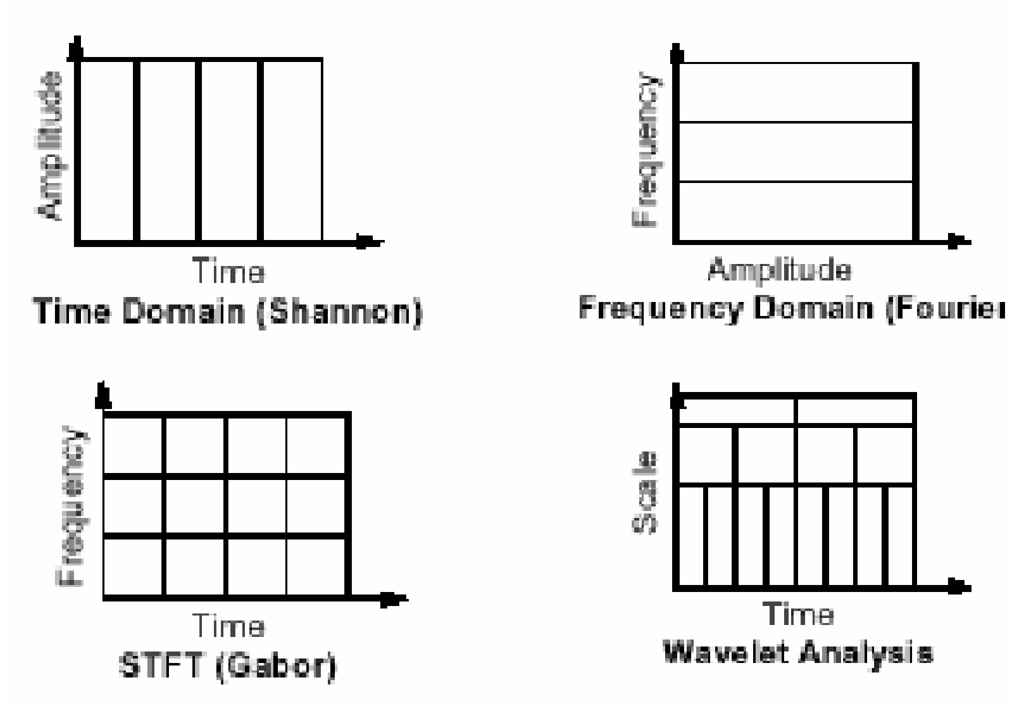


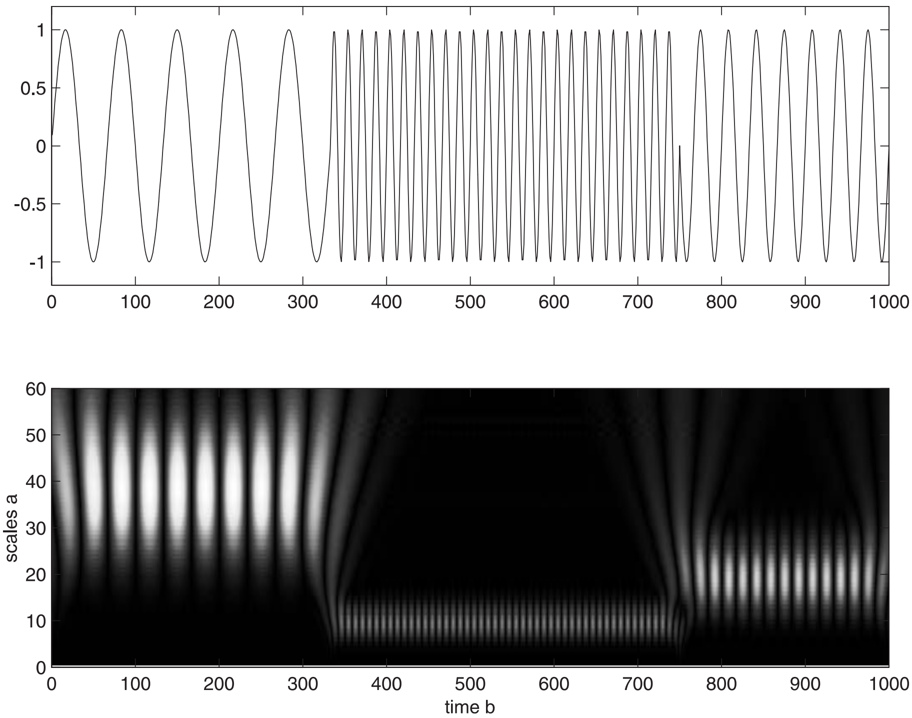
AMPLITUDE
4
3
2
1
0
150
100
50
0
SCALE
0
20
40
60
80
100
TRANSLATION
THE WAVELET TUTORIAL
All Rights Reserved, Robi Polikar, Ames, Iowa, 1996

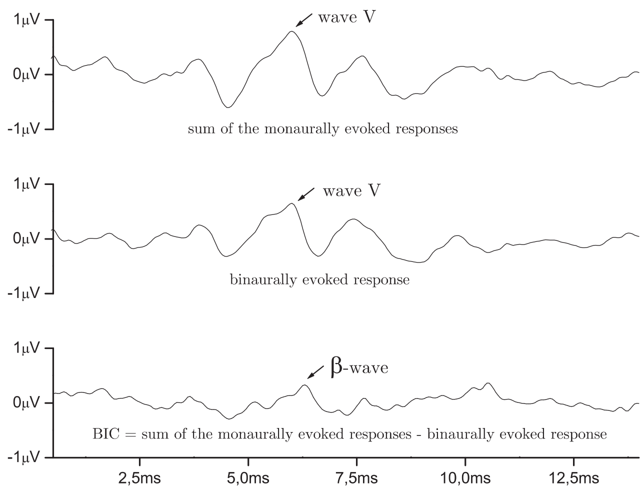


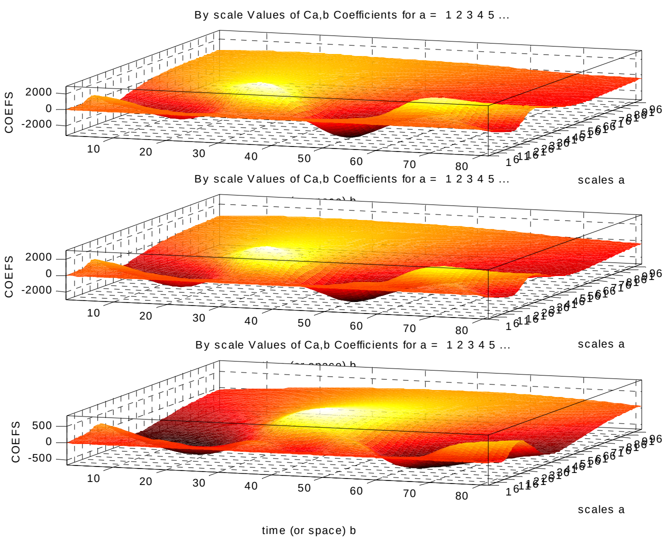
-5
-4
-3
-2
-1
0
1
2
3
4
5


Waveform
Wavelet
Coe. of WT
Data 1
Data 2
Data 3
Data 4
Mexican
Morlte R.
Morlet I.
Morlet
T-F Dis.
Design:
Lingsong He


Amplitude
Time
Time Domain (Shannon)
Frequency
Amplitude
Frequency Domain (Fourier
Frequency
Time
STFT (Gabor)
Scale
Time
Wavelet Analysis


scales a
time b


wave V
sum of the monaurally evoked responses
wave V
binaurally evoked response
β-wave
BIC = sum of the monaurally evoked responses - binaurally evoked response
1μV
0μV
-1μV
2,5ms
5,0ms
7,5ms
10,0ms
12,5ms

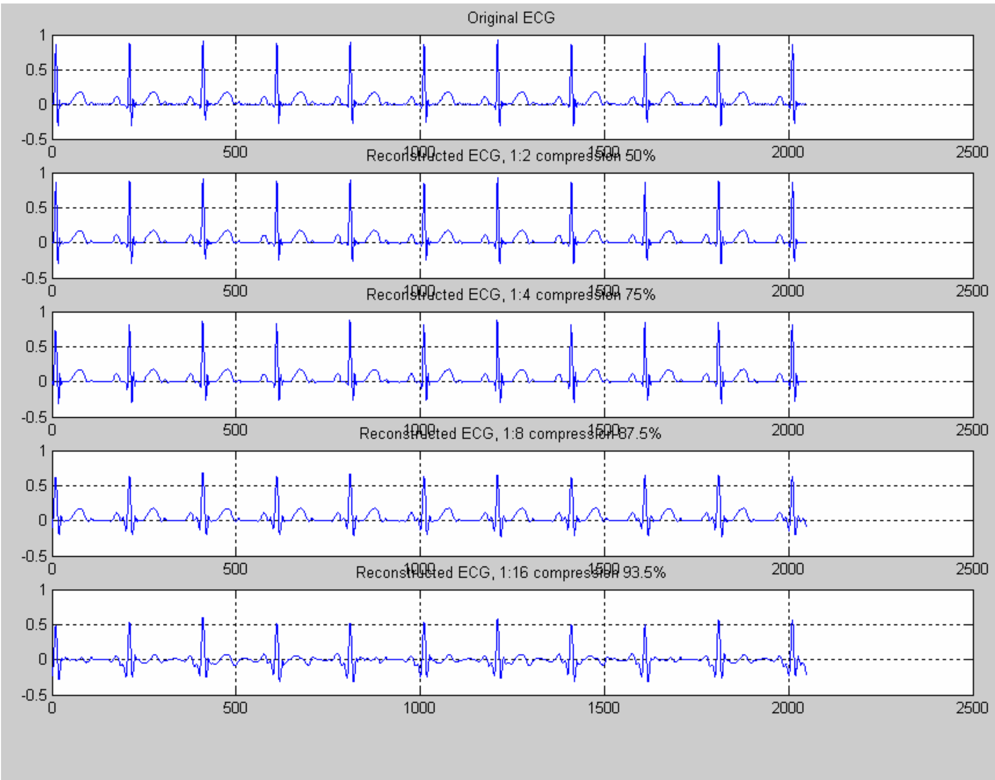

By scale Values of Ca,b Coefficients for a = 1 2 3 4 5 ...
COEFS
scales a
time (or space) b

Discrete Wavelet Transform

- CWT computed by computers is really not CWT, it is a discretized version of the CWT.
- The resolution of the time-frequency grid can be controlled (within Heisenberg's inequality), can be controlled by time and scale step sizes.
- Often this results in a very **redundant** representation
- How to discretize the continuous time-frequency plane, so that the representation is non-redundant?
  - **Sample the time-frequency plane on a dyadic (octave) grid**

SEE BLACKBOARD

ECG- Compression



Multiresolution