Computational Diagnostics & Biocybernetics Unit

BMT 801 : Biomedizinische Signal- und Bildverarbeitung EF1 („adhoc quick & dirty Einführung“)

Prof. Dr. Dr. Daniel J. Strauss

- Signalkategorien

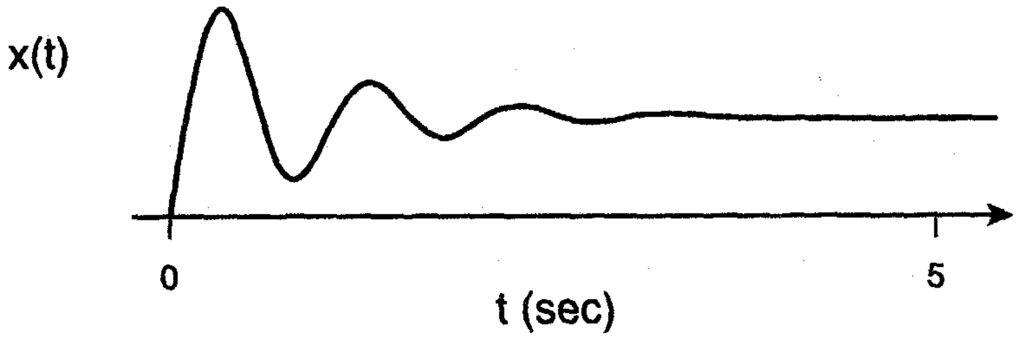
  - Amplituden- und Zeitkontinuierlich.

  - Amplitudenkontinuierlich und Zeitdiskret.

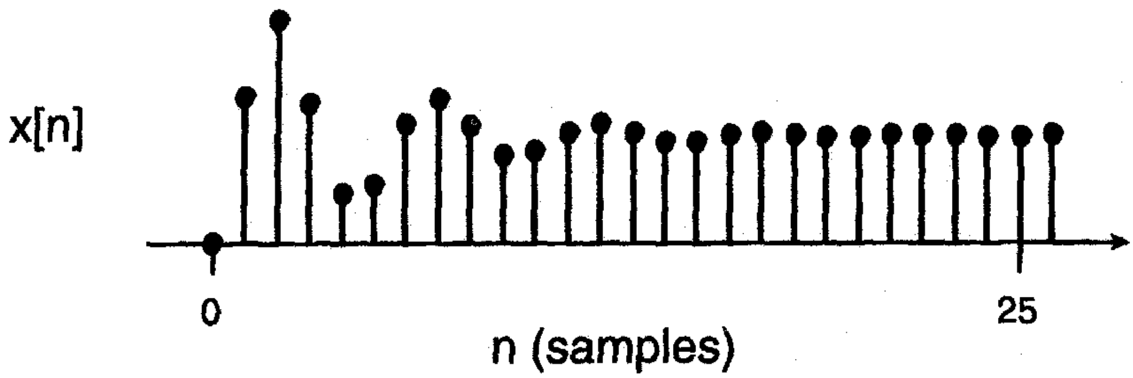
  - Amplitudendiskret und Zeitkontinuierlich.

  - Amplituden- und Zeitdiskret

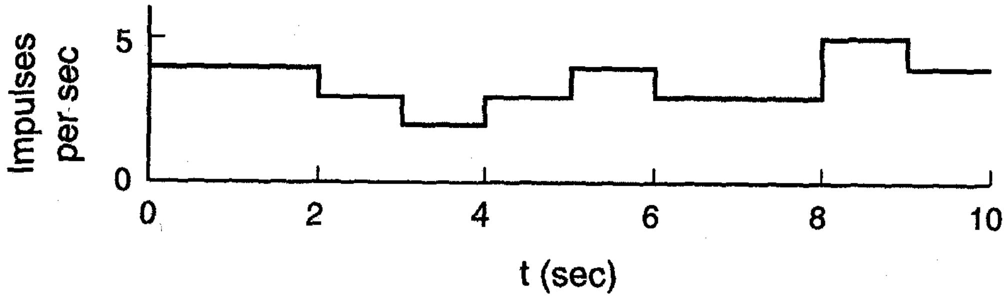
Amplituden- und Zeitkontinuierlich.



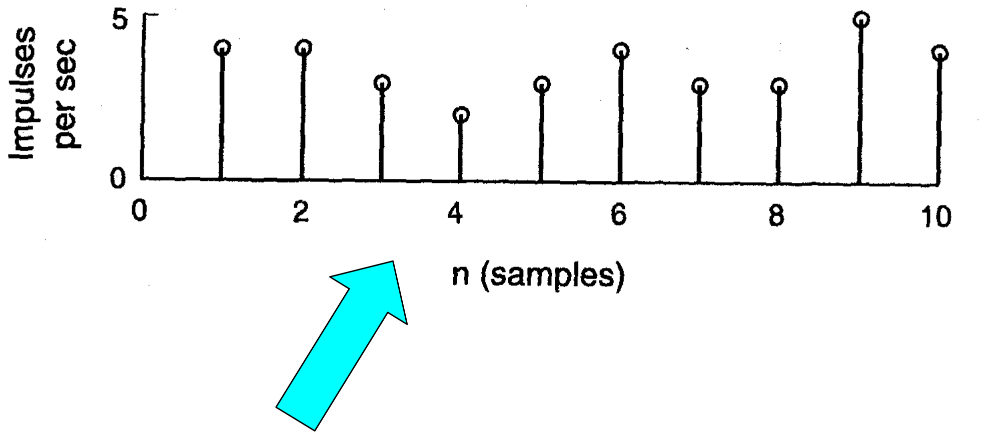
Amplitudenkontinuierlich und Zeitdiskret.



Amplitudendiskret und Zeitkontinuierlich.

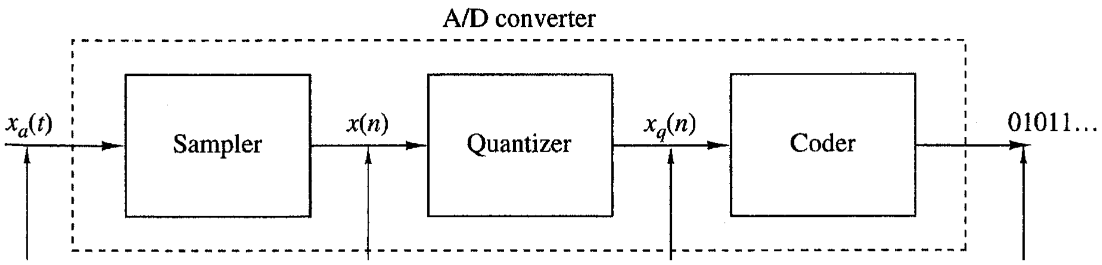


Amplituden- und Zeitdiskret



Dies wird unser Fokus in der Vorlesung sein, warum?

Analog- und Digitalwandler A/D-Konverter



Amplituden- und Zeitkontinuierlich

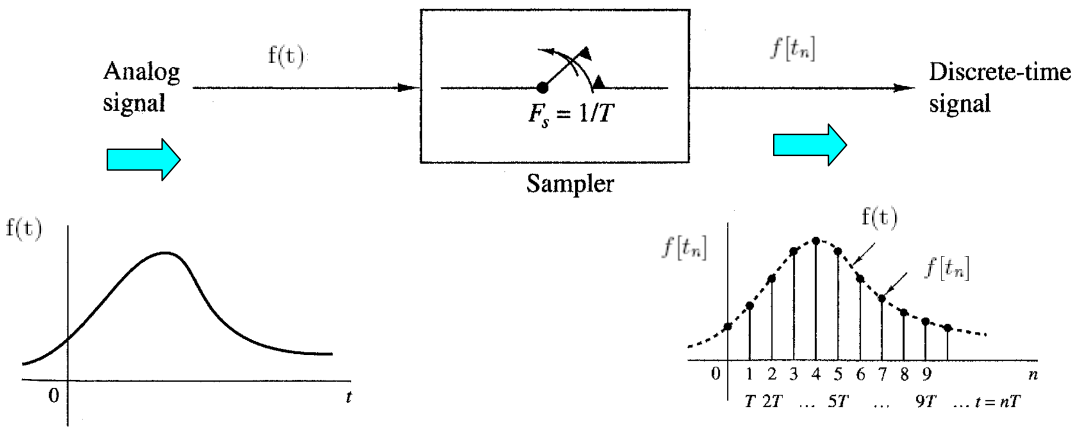
Amplitudenkontinuierlich und Zeitdiskret

Amplituden- und Zeitdiskret

Codiertes Signal 01100010011

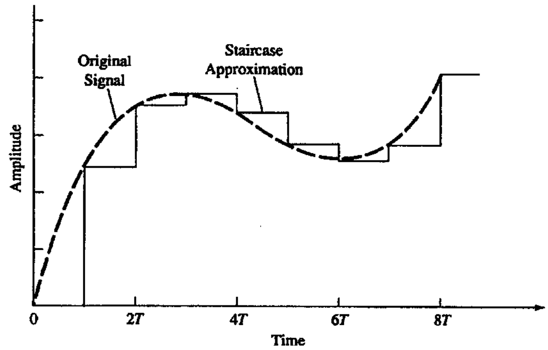
Sampler

- einheitliche Abtastung mit konstanter Frequenz
- das Öffnen und Schließen des „Schalters" erfolgt mit der Abtastfrequenz fs

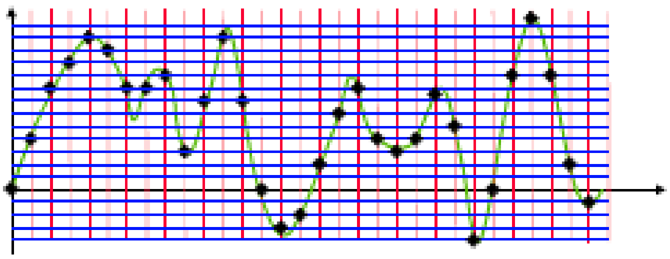


Quantisierung

- Approximiert kontinuierliches Signal in eine endliche Anzahl von „äquidistanten Intervallen".
- Intervall hat konstanten Wert
- Kleinster Schritt: Messbereich/Anzahl der Intervalle.



Sampling und Quantisierung



Beispiel: USBamp

Messbereich $U_{ss} = \pm 250\ mV$

Auflösung $24\ Bit\ =\ 2^{24}\ =\ 16,8\ 10^6$

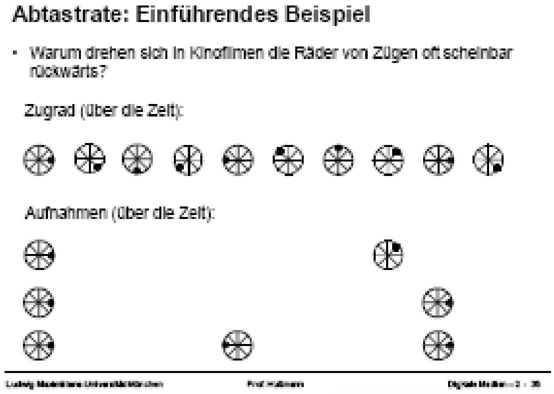
kleinstes Quantil $Q\ =\ \frac{U_{ss}}{N-1}\ =\ \frac{0,5\ V}{2^{24}-1}\ \approx\ 30\ nV$

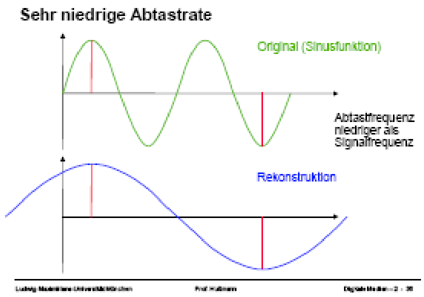
Abtastfrequenz von 16 Hz bis 38 kHz

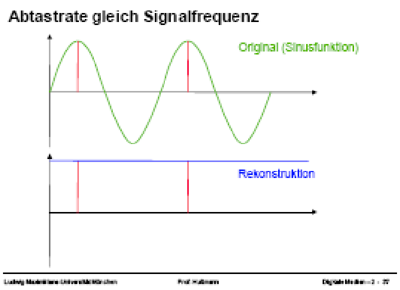
Sampling und Quantisierung

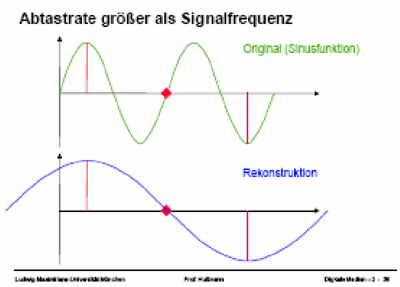
Fehlerquellen

zu geringe Samplesfrequenz





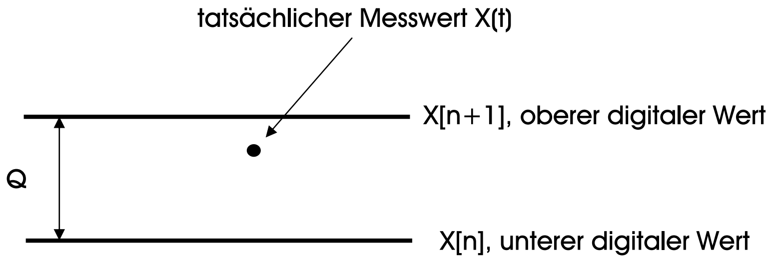




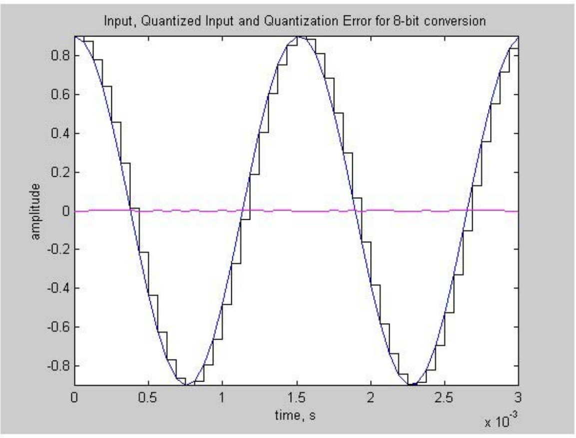
Sampling und Quantisierung

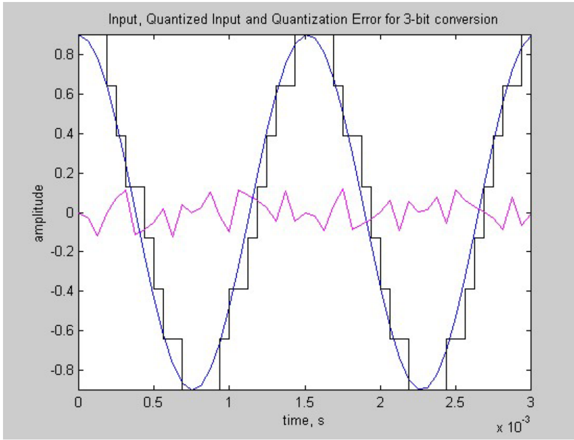
Fehlerquellen

Quantisierungsfehler



- minimaler Fehler 0, maximaler Fehler Q.
- mittlerer Fehler Q / 2.
- Fehler der Quantisierung wird auch „digitales Rauschen“ genannt.





Abtasttheorem

Das Spektrum einer Folge X[n] ergibt sich zu:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

mit $e^{-j\omega t_0} = \mathcal{F}\left(\delta(t - t_0)\right)$

erhält man

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, \mathcal{F}\left(\delta[t - nT]\right)$$

°

°

°

das Ergebnis

$$X(\omega) = \omega_T \cdot \sum_{n=-\infty}^{+\infty} X(\omega - \omega_T)$$

D.h. das Spektrum X(ω) der Folge X[n] ergibt sich aus der Überlagerung der jeweils um ein ganzzahliges Vielfaches der Periode 1/T verschobenen Spektren des analogen Signals.

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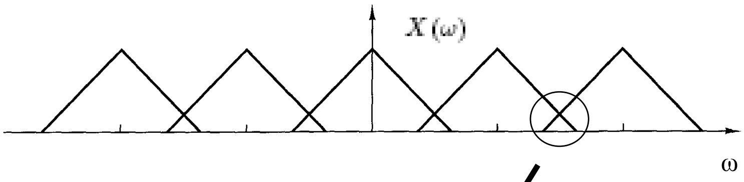
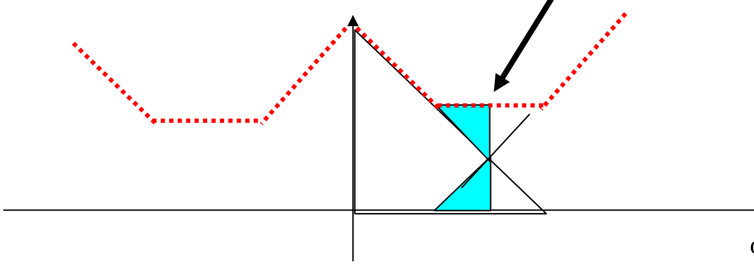
Abtasttheorem

Ein bandbegrenztes digitales Signal, welches mit einer zu geringen Abtastrate gemessen wurde, hat im Frequenzbereich eine Überlappung der Spektren.





Diese Überlappung führt bei der Signalsynthese zu einem Verlust von Informationen.

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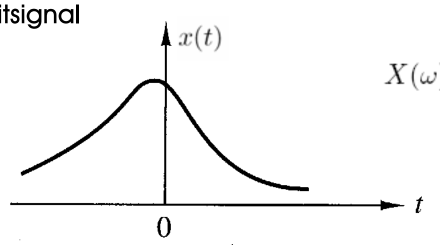
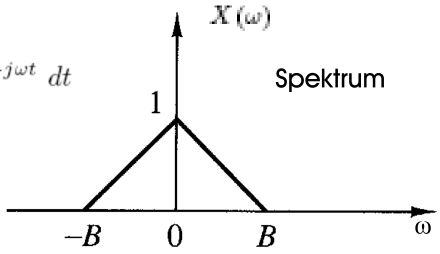
Abtasttheorem

Zeitsignal



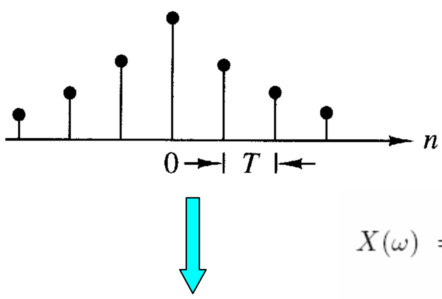
$$X(\omega) = \int_{-\omega}^{+\omega} x(t)\, e^{-j\omega t}\, dt$$

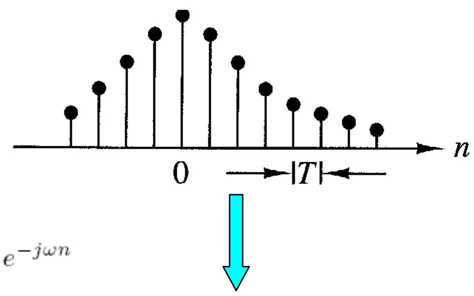


Spektrum

geringe Abtastrate

hohe Abtastrate





$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

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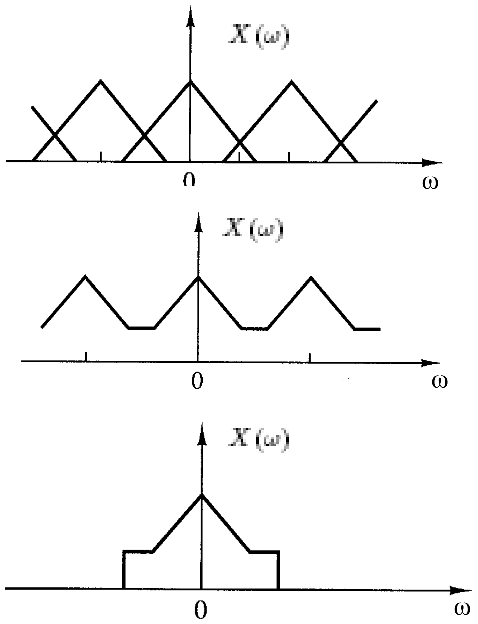
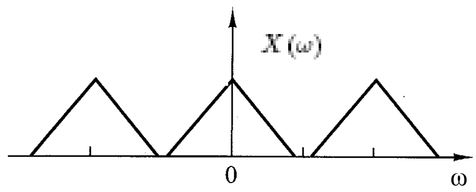
Abtasttheorem

geringe Abtastrate

hohe Abtastrate





$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

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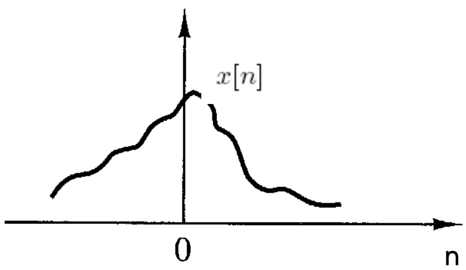
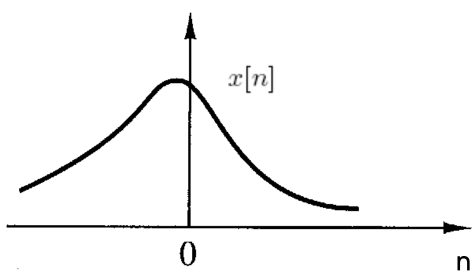
Abtasttheorem

geringe Abtastrate

hohe Abtastrate





Die Abtastung im Zeitbereich führt zu einer Periodisierung im Frequenzbereich (und umgekehrt). Eine Überlappung der Frequenzspektren kann dadurch verhindert werden, dass das bandbegrenzte Eingangssignal mit einer deutlich über der Grenzfrequenz befindlichen Samplefrequenz abgetastet wird.

Nach Shannon muss die Samplefrequenz min. doppelt so hoch sein, wie die Grenzfrequenz des Eingangssignals.

$$f_s > 2 \cdot f_g$$

• Signaltypen

- Deterministische Signale

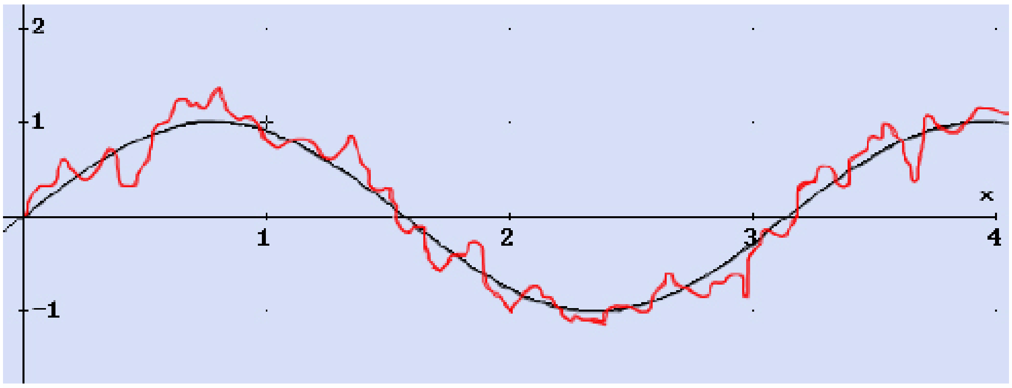
- Stochastische Signale

- Chaotische Signale

- Fraktale Signale

$$x(t) = a(t) + s(t)$$

x(t) = gemessenes Signal
a(t) = deterministischer Anteil
s(t) = Rauschanteil/stochastischer Signalanteil/Störsignal



FOURIER ANALYSIS

PART 1:

Frequency Analysis & Fourier Series

Slides are partly prepared by Maria Elena Angoletta
CERN (European Organization for Nuclear Research), Switzerland

TOPICS

1. Frequency analysis: a powerful tool
2. A tour of Fourier Transforms
3. Continuous Fourier Series (FS)
4. Discrete Fourier Series (DFS)

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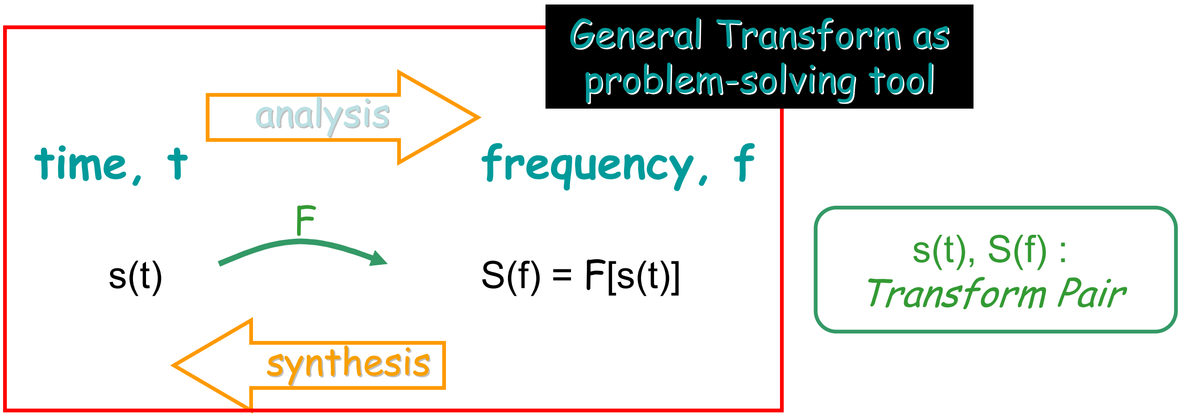
Frequency analysis: why?

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns. (PDE).
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.



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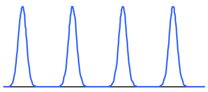
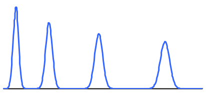
Fourier analysis - applications

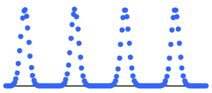
Applications wide ranging and ever present in modern life

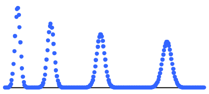
- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated "devices", biometry, …).

Fourier analysis - tools









Input Time Signal			Frequency spectrum	
Continuous	Periodic *(period T)*	**FS**	Discrete	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$
Continuous	Aperiodic	**FT**	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$
Discrete	Periodic *(period T)*	**DFS****	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$
Discrete	Aperiodic	**DTFT**	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$
Discrete	Aperiodic	**DFT****	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$

Note: $j = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples

** *Calculated via FFT*

A little history

- Astronomic predictions by Babylonians/Egyptians likely via trigonometric sums.

- **1669**: Newton stumbles upon light spectra (*specter* = ghost) but fails to recognise "frequency" concept (*corpuscular* theory of light, & no waves).

- **18th century**: two outstanding problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation). **BUT** peers' consensus is that sum of sinusoids *only* represents smooth curves. Big blow to utility of such sums for all but Fourier ...

- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines. Strong criticism by peers blocks publication. Work published, 1822 *("Theorie Analytique de la chaleur")*.

A little history -2

- **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

CONTINUOUS

→ Fourier extends the analysis to arbitrary function (Fourier Transform).

→ Dirichlet, Poisson, Riemann, Lebesgue address FS convergence.

→ Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

DISCRETE: Fast calculation methods (FFT)

→ **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).

→ **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).

→ Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).

→ FFT algorithm refined & modified for most computer platforms.

Fourier Series (FS)

A <u>periodic</u> function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

synthesis

$$s(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k \cdot \cos(k\,\omega\, t) - b_k \cdot \sin(k\,\omega\, t) \right]$$

For all t but discontinuities

a_0, a_k, b_k : Fourier coefficients.

k: harmonic number,

T: period, $\omega = 2\pi/T$

analysis

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)\,dt$$

(signal average over a period, i.e. DC term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\,\omega\, t)\,dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\,\omega\, t)\,dt$$

* see next slide

FS convergence

Dirichlet conditions

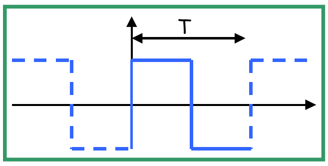
In any period:

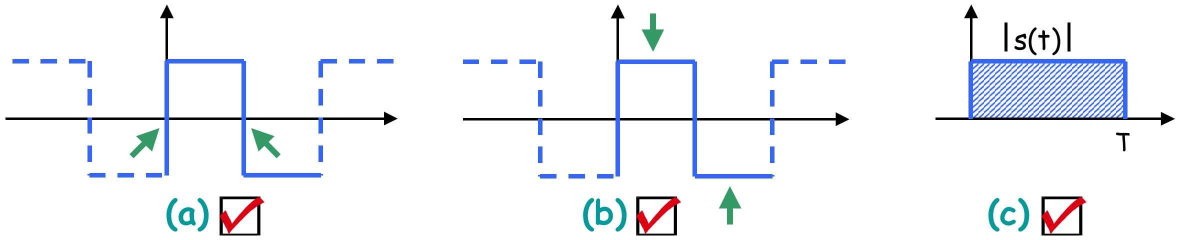
(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

(c) s(t) absolutely integrable , $\int_0^T |s(t)|\,dt < \infty$

Example: square wave





$x \le y$, then $f(x) \le f(y)$

FS analysis - 1

FS of odd* function: square wave.

$T = 2\pi \quad \Rightarrow \omega = 1$

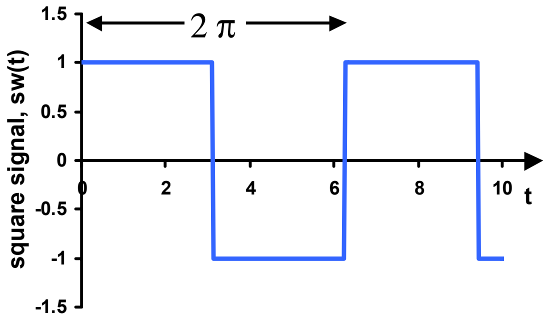
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\cos kt\,dt - \int_{\pi}^{2\pi}\cos kt\,dt\right\} = 0$$ *(odd function)*

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\sin kt\,dt - \int_{\pi}^{2\pi}\sin kt\,dt\right\} = \dots = \frac{2}{k\cdot\pi}\cdot\{1-\cos k\pi\} =$$

$$= \begin{cases} \dfrac{4}{k\cdot\pi} & , \text{k odd} \\ 0 & , \text{k even} \end{cases}$$

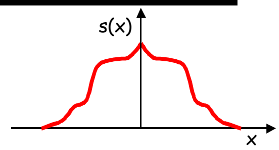
$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \dots$$



* Even & Odd functions

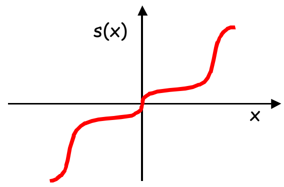
Even :

s(-x) = s(x)



Odd :

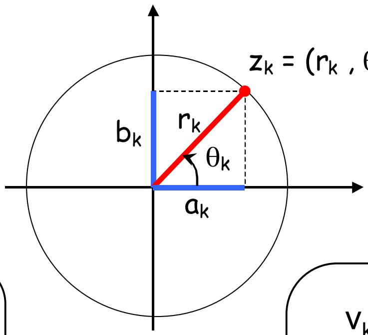
s(-x) = -s(x)



FS analysis - 2

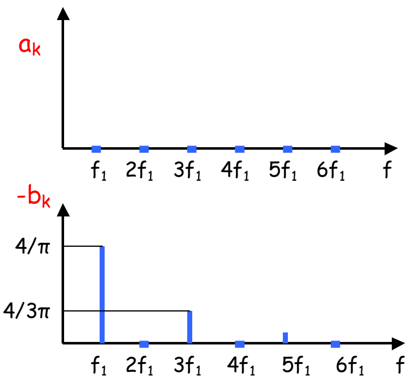
Fourier spectrum representations

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$



[redacted]

$v_k = a_k\cos(\omega k\ t) - b_k\sin(\omega k\ t)$



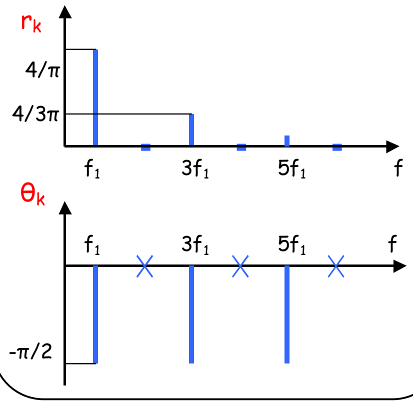
r_K
θ_K [redacted]

$f_k = k\ \omega/2\pi$

Fourier spectrum of square-wave.

[redacted]

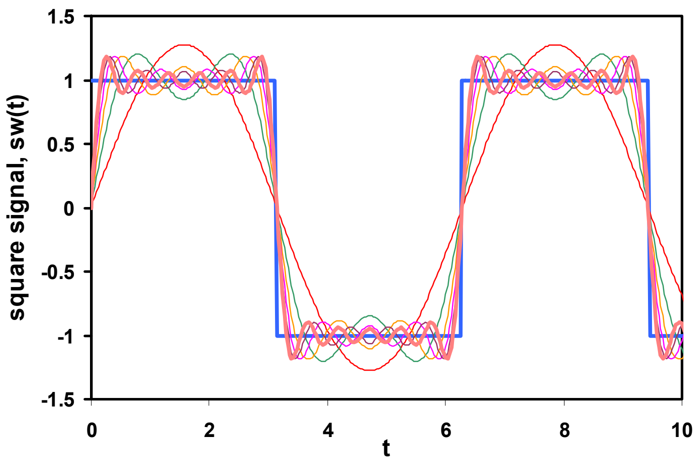
$v_k = r_k \cos(\omega k\ t + \theta_k)$



FS synthesis

Square wave reconstruction from spectral terms

$$sw_9(t) = \sum_{k=1}^{9}\left[b_k \sin(kt)\right]$$



Convergence may be slow (~1/k) - ideally need infinite terms.

Practically, series truncated when remainder below computer tolerance (⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

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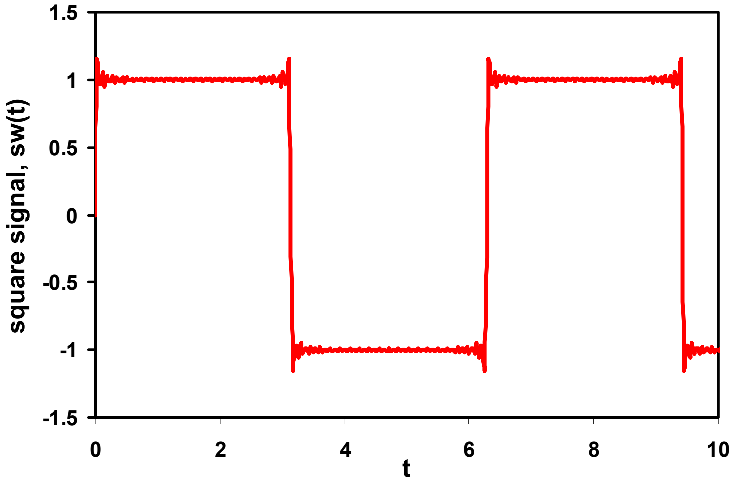
Gibbs phenomenon

Overshoot exist @ each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} \left[-b_k \cdot \sin(kt) \right]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude. Just a minor annoyance.

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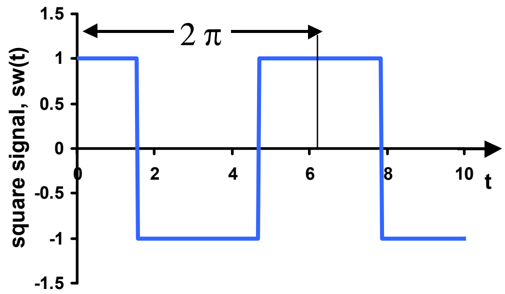
FS time shifting

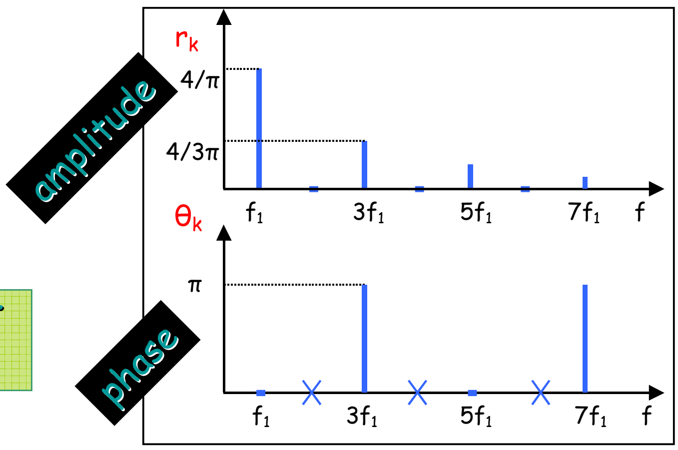
FS of even function: $\pi/2$-advanced square-wave

$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \dfrac{4}{k \cdot \pi}, & k \text{ odd}, \quad k = 1, 5, 9... \\ -\dfrac{4}{k \cdot \pi}, & k \text{ odd}, \quad k = 3, 7, 11... \\ 0, & k \text{ even}. \end{cases}$$

$-b_k = 0$ *(even function)*





Note: amplitudes unchanged ***BUT*** phases advance by $k \cdot \pi/2$.

Complex FS

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - j\cdot\sin(t)$ ⟹ "phasor" $\cos(t) = \frac{e^{jt}+e^{-jt}}{2}$ $\sin(t) = \frac{e^{jt}-e^{-jt}}{2\cdot j}$

analysis

$$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$$

synthesis

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

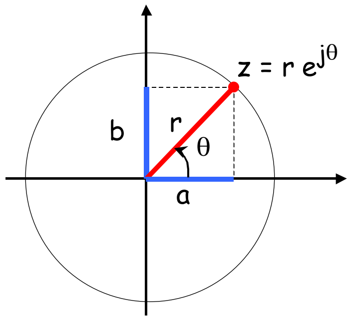
Complex form of FS (Laplace 1782). Harmonics c_k separated by $\Delta f = 1/T$ on frequency plot.

Note: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

$$c_0 = a_0$$

$$c_k = \frac{1}{2}\cdot(a_k + j\cdot b_k) = \frac{1}{2}\cdot(a_{-k} - j\cdot b_{-k})$$



$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$

FS properties

	Time	Frequency
Homogeneity	$a\cdot s(t)$	$a\cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k)+U(k)$
Linearity	$a\cdot s(t) + b\cdot u(t)$	$a\cdot S(k)+b\cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t)\cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T}\cdot\int_0^T s(t-\bar{t})\cdot u(\bar{t})d\bar{t}$	$S(k)\cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j\frac{2\pi k\cdot\bar{t}}{T}}\cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi m t}{T}}\cdot s(t)$	$S(k - m)$

FS - power

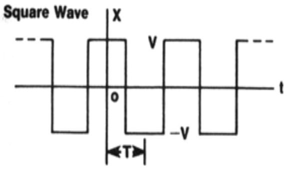
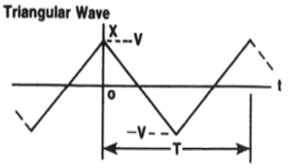
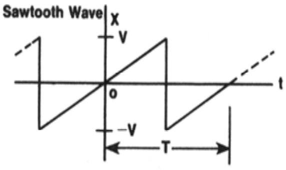
Average power W : $W = \frac{1}{T}\int_{o}^{T} |\, s(t) \,|^2 \, dt \equiv s(t) \otimes s(t)$

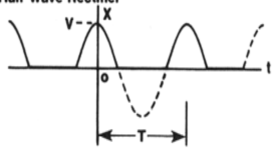
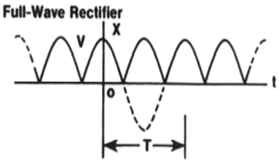
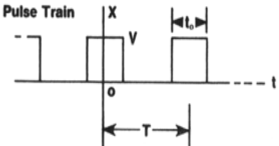
Parseval's Theorem

$$W = \sum_{k=-\infty}^{\infty} |\, c_k \,|^2 = a_0{}^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k{}^2 + b_k{}^2 \right)$$

- FS convergence ~1/k
  - $\Rightarrow$ *lower frequency terms*
  - *W_k = $|c_k|^2$ carry most power.*
- W_k vs. ω_k: Power density spectrum.

FS of main waveforms

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Square Wave 	$x(t) = \frac{4V}{\pi}(\cos \omega_o t - \frac{1}{3}\cos 3\omega_o t + \frac{1}{5}\cos 5\omega_o t - \frac{1}{7}\cos 7\omega_o t + \ldots)$
Triangular Wave 	$x(t) = \frac{8V}{\pi^2}(\cos \omega_o t + \frac{1}{9}\cos 3\omega_o t + \frac{1}{25}\cos 5\omega_o t + \ldots)$
Sawtooth Wave 	$x(t) = \frac{2V}{\pi}(\sin \omega_o t - \frac{1}{2}\sin 2\omega_o t + \frac{1}{3}\sin 3\omega_o t - \frac{1}{4}\sin 4\omega_o t + \ldots)$

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Half-Wave Rectifier 	$x(t) = \frac{V}{\pi}(1 + \frac{\pi}{2}\cos \omega_o t + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots)$ n even
Full-Wave Rectifier 	$x(t) = \frac{2V}{\pi}(1 + \frac{2}{3}\cos 2\omega_o t - \frac{2}{15}\cos 4\omega_o t + \frac{2}{35}\cos 6\omega_o t - \ldots \ldots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t \ldots)$ n even
Pulse Train  1754-16 $k = t_o/T$	$x(t) = V[k + \frac{2}{\pi}(\sin k\pi \cos \omega_o t + \frac{1}{2}\sin 2k\pi \cos 2\omega_o t + \ldots \ldots + \frac{1}{n}\sin nk\pi \cos n\omega_o t + \ldots)]$ $k = t_o/T$

Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

DFS defined as:

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ same period N

i.e. time periodicity propagates to frequencies!

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Synthesis: finite sum $\Leftarrow$ band-limited s[n]

DFS generate periodic c_k with same signal period

Orthogonality in DFS

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

↑ Kronecker's delta

N consecutive samples of s[n] completely describe s in time or frequency domains.

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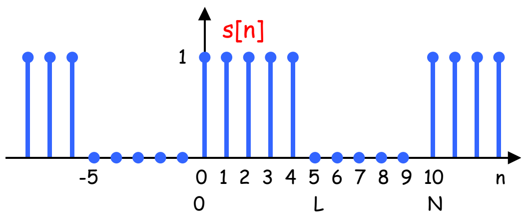
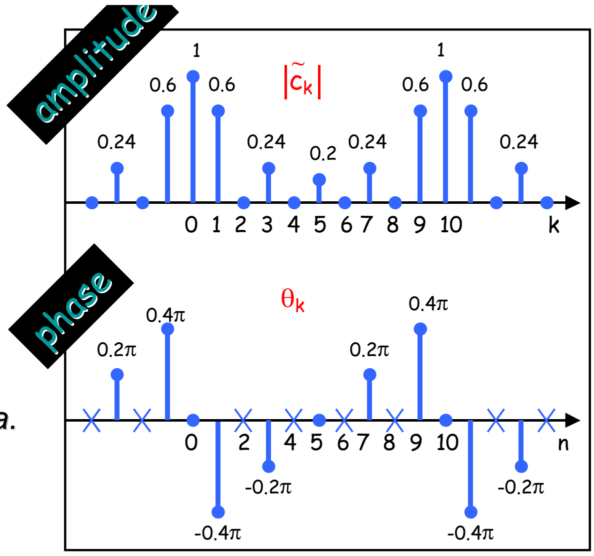
DFS analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\[2ex] \dfrac{e^{-j\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\frac{\pi\, kL}{N}\right)}{\sin\left(\frac{\pi\, k}{N}\right)}, & \text{otherwise} \end{cases}$$





Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function

DFS properties

	Time	Frequency
Homogeneity	$a \cdot s[n]$	$a \cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication *	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution *	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-j\frac{2\pi\, k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi\, h\, t}{T}} \cdot s[n]$	$S(k - h)$

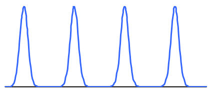
** Explained in next week's lecture*

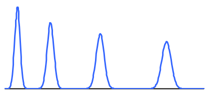
TOPICS

1. **Infinite Fourier Transform (FT)**
2. **FT & generalised impulse**
3. **Uncertainty principle**
4. **Discrete Time Fourier Transform (DTFT)**
5. **Discrete Fourier Transform (DFT)**
6. **Comparing signal by DFS, DTFT & DFT**

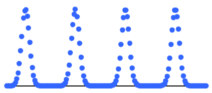
Fourier analysis - tools

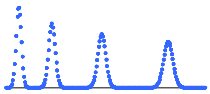
Input Time Signal — **Frequency spectrum**





Continuous
- Periodic *(period T)* — **FS** — Discrete — $c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$
- Aperiodic — **FT** — Continuous — $S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$





Discrete
- Periodic *(period T)* — **DFS** — Discrete — $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$
- Aperiodic
  - **DTFT** — Continuous — $S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$
  - **DFT** — Discrete — $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$

Note: $j = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n] = s(t_n)$, N = No. of samples

Fourier Integral (FI)

Fourier analysis tools for aperiodic signals.

Fourier Integral Theorem

Any aperiodic signal s(t) can be expressed as a Fourier integral if s(t) piecewise smooth(1) in any finite interval (-L,L) and absolute integrable(2).

$$s(t) = \int_0^{+\infty} \{A(\omega) \cdot \cos(\omega t) + B(\omega) \cdot \sin(\omega t)\} d\omega \quad (3)$$

(1) s(t) continuous, s'(t) monotonic

(2) $\int_{-\infty}^{+\infty} |s(t)| dt < +\infty$

(3) $A(\omega) = \frac{1}{\pi} \cdot \int_{-\infty}^{+\infty} s(t) \cos(\omega t) dt \qquad B(\omega) = \frac{1}{\pi} \cdot \int_{-\infty}^{+\infty} s(t) \sin(\omega t) dt$

Real-to-complex link

$$S(\omega) = \pi \cdot [A(\omega) - j \cdot B(\omega)]$$

Complex form

analysis

$$S(\omega) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t} dt$$

synthesis

$$s(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S(\omega) \cdot e^{j\omega t} d\omega$$

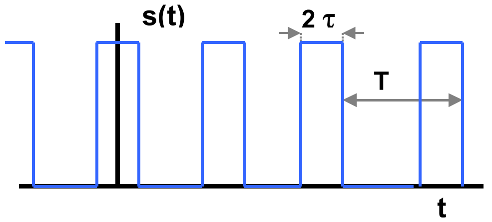
Fourier Transform (Pair) - FT

Let's summarise a little

Signal →	Periodic	Aperiodic
Domain ↓		
Time	FS	FI
Frequency – real	$a_k,\ b_k$	$A(\omega),\ B(\omega)$
Frequency – complex	c_k	$C(\omega)$ → FT

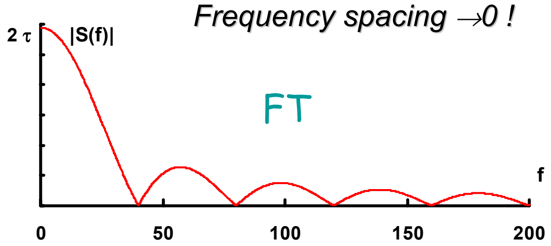
From FS to FT

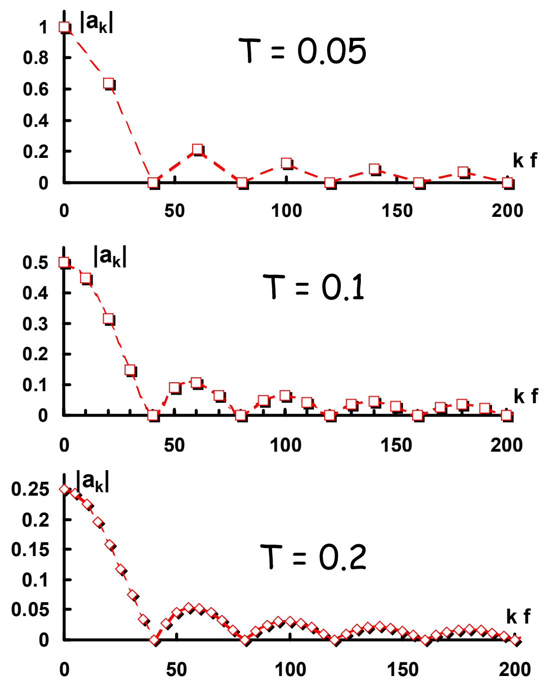
FS moves to FT as period T increases: ➡ continuous spectrum



Pulse train, width 2 τ = 0.025

Frequency spacing →0 !





Note: $|a_k| \to 2\,a_0$ as $k \to 0 \Rightarrow 2\,a_0$ is plotted at k=0

Getting FS from FT

FS defined

$$c_k = \frac{1}{T}\cdot \int_{-T/2}^{T/2} s(t)\cdot e^{-jk\omega t}\,dt$$

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

$\Delta f = \Delta\omega/(2\pi) = 1/T$

frequency spacing

1

$$c_k = \Delta f \cdot \int_{-T/2}^{T/2} s(t)\cdot e^{-j\,\omega_k t}\,dt \equiv c_{\omega_k}$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} c_{\omega_k} \cdot e^{j\,\omega_k t}$$

2

$$\Gamma_{\omega_k} \equiv \frac{c_{\omega_k}}{\Delta f} = \int_{-T/2}^{T/2} s(t)\cdot e^{-j\,\omega_k t}\,dt$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} \Gamma_{\omega_k} \cdot e^{j\,\omega_k t}\,\Delta f$$

As $\Delta f \to 0$, replace Δf, ω_K, $\sum_{\omega_k=-\infty}^{\infty}$

by df, $2\pi f$, $\int_{-\infty}^{+\infty}$

FT defined

$$S(f) = \lim_{\Delta f\to 0} \Gamma_{\omega_k} = \int_{-\infty}^{+\infty} s(t)\cdot e^{-j\omega t}\,dt$$

$$s(t) = \int_{-\infty}^{+\infty} S(f)\cdot e^{j2\pi ft}\,df$$

FT properties

	Time	Frequency
Linearity	$a\cdot s(t) + b\cdot u(t)$	$a\cdot S(f)+b\cdot U(f)$
Multiplication	$s(t)\cdot u(t)$	$\int_{-\infty}^{+\infty} S(f-\bar{f})\cdot U(\bar{f})\,d\bar{f}$
Convolution	$\int_{-\infty}^{+\infty} s(t-\bar{t})\cdot u(\bar{t})\,d\bar{t}$	$S(f)\cdot U(f)$
Time shifting	$s(t-\bar{t})$	$e^{-j\,2\pi\,f\,\bar{t}}\cdot S(f)$
Frequency shifting	$e^{+j2\pi\bar{f}}\cdot s(t)$	$S(f-\bar{f})$
Time reversal	$s(-t)$	$S(-f)$
Differentiation	$\frac{ds(t)}{dt}$	$j2\pi f\,S(f)$
Parseval's identity	$\int h(t)\,g^*(t)\,dt = \int H(f)\,G^*(f)\,df$	
Integration	$\int_{-\infty}^{t} s(u)\,du$	$S(f)/(j2\pi f\,)$
Energy & Parseval's (E is t-to-f invariant)	$E = \int_{-\infty}^{+\infty} \lvert s(t)\rvert^2 dt = \int_{-\infty}^{+\infty} \lvert S(f)\rvert^2 df$	

FT - uncertainty principle

Bandwidth Theorem

For effective duration Δt & bandwidth Δf

$$\exists \gamma > 0 \quad \underbrace{\Delta t \cdot \Delta f}_{\text{uncertainty product}} \geq \gamma$$

Fourier uncertainty principle

For Energy Signals:

$E=\int |s(t)|^2 dt = \int |S(f)|^2 df < \infty$

Define mean values

$$\bar{t}^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} t \cdot |s(t)|^2 \, dt \qquad \bar{f}^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} f \cdot |S(f)|^2 \, df$$

Define std. dev.

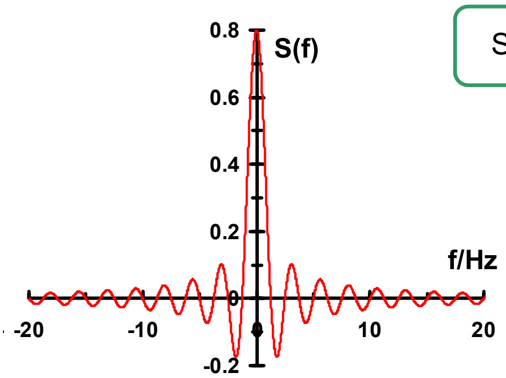
$$\Delta t^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} (t - \bar{t}) \cdot |s(t)|^2 dt \qquad \Delta f^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} (f - \bar{f}) \cdot |S(f)|^2 df$$

$\Rightarrow \Delta t \cdot \Delta f \geq 1/4\pi$

Implications

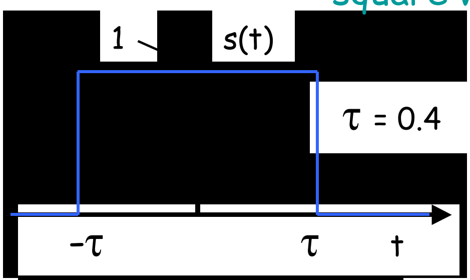
- Limited accuracy on simultaneous observation of s(t) & S(f).
- Good time resolution (small Δt) requires large bandwidth Δf & vice-versa.

FT - example



$S(f) = 2\tau\, s_{MAX}\, \mathrm{sync}(2f\tau)$

FT of 2τ-wide square window



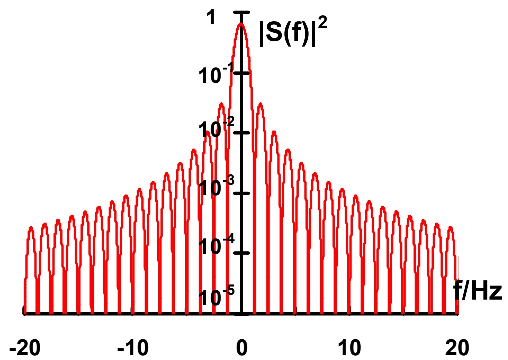
Fourier uncertainty

Choose

$\Delta t = |\int s(t)/s(0)\, dt| = 2\tau$,

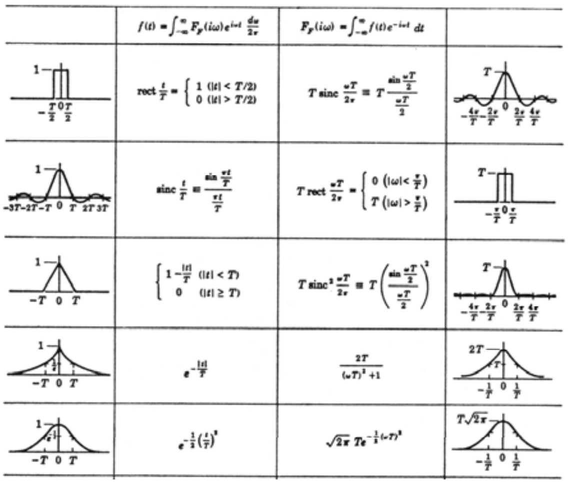
$\Delta f = |\int S(f)/S(0)\, df| = 1/(2\tau)$ = half distance btwn first 2 zeroes ($f_{1,-1} = \pm 1/2\tau$) of S(f)

then: $\Delta t \cdot \Delta f = 1$

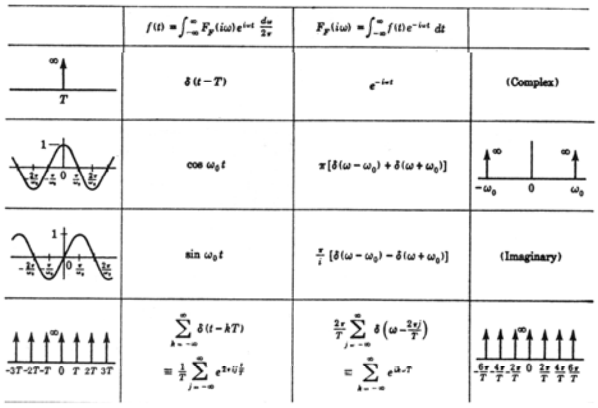


Power Spectral Density (PSD) vs. frequency f plot.
Note: Phases unimportant!

FT of main waveforms



	$f(t) = \int_{-\infty}^{\infty} F_p(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_p(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	
	$\mathrm{rect}\,\frac{t}{T} = \begin{cases} 1 & (\lvert t \rvert < T/2) \\ 0 & (\lvert t \rvert > T/2) \end{cases}$	$T\,\mathrm{sinc}\,\frac{\omega T}{2\pi} \equiv T \frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\mathrm{sinc}\,\frac{t}{T} \equiv \frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T\,\mathrm{rect}\,\frac{\omega T}{2\pi} = \begin{cases} 0 & (\lvert\omega\rvert < \frac{\pi}{T}) \\ T & (\lvert\omega\rvert > \frac{\pi}{T}) \end{cases}$	
	$\begin{cases} 1 - \frac{\lvert t \rvert}{T} & (\lvert t \rvert < T) \\ 0 & (\lvert t \rvert \geq T) \end{cases}$	$T\,\mathrm{sinc}^2\,\frac{\omega T}{2\pi} \equiv T \left(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$	
	$e^{-\frac{\lvert t \rvert}{T}}$	$\frac{2T}{(\omega T)^2 + 1}$	
	$e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2}$	$\sqrt{2\pi}\, T e^{-\frac{1}{2}(\omega T)^2}$	



	$f(t) = \int_{-\infty}^{\infty} F_p(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_p(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	
	$\delta(t - T)$	$e^{-i\omega T}$	(Complex)
	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
	$\sin \omega_0 t$	$\frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	(Imaginary)
	$\sum_{k=-\infty}^{\infty} \delta(t - kT) \equiv \frac{1}{T} \sum_{j=-\infty}^{\infty} e^{2\pi i j \frac{t}{T}}$	$\frac{2\pi}{T} \sum_{j=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi j}{T}\right) \equiv \sum_{k=-\infty}^{\infty} e^{ik\omega T}$	

Discrete Time FT (DTFT)

DTFT defined as:

analysis

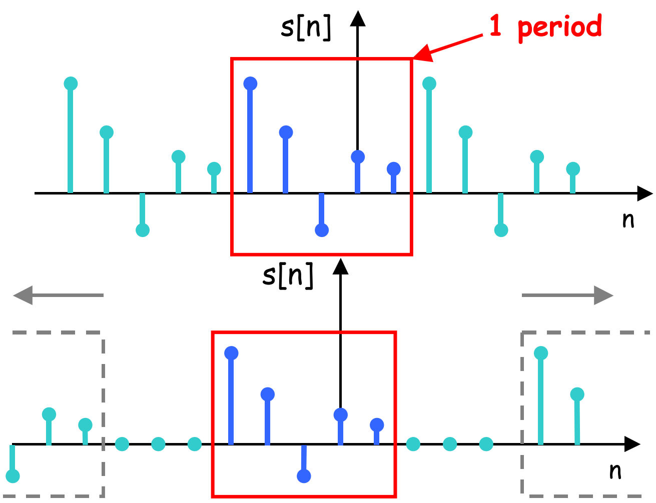
$$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j 2\pi f n}$$

synthesis

$$s[n] = \frac{1}{2\pi} \cdot \int_{0}^{2\pi} S(f) e^{j 2\pi f n} df$$

Note: continuous frequency domain! (frequency density function)

Holds for aperiodic signals

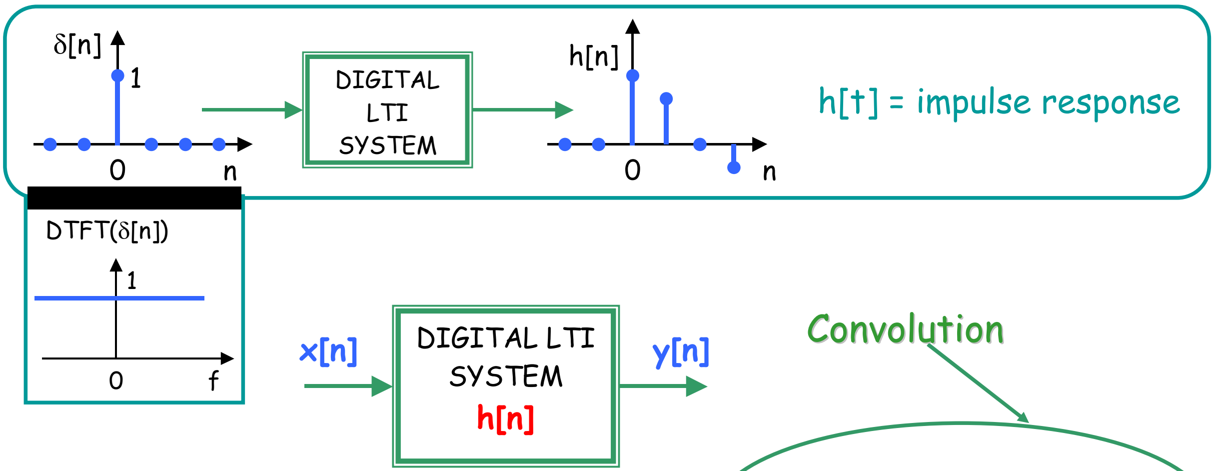


Obtained from DFS as $N \to \infty$

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j \frac{2\pi k n}{N}}$$

DTFT - convolution

Digital Linear Time Invariant system: obeys superposition principle.



$$y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[n-m] \cdot h[m]$$

$x[n] \Updownarrow X(f)$, $h[n] \Updownarrow H(f)$, $y[n] \Updownarrow Y(f) = X(f) \cdot H(f)$

kernel of FT as eigenfunction of LTI systems – see blackboard

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Discrete FT (DFT)

DFT defined as:

analysis

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$$

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Frequency resolution

Analysis frequencies f_m

$$f_m = \frac{m \cdot f_S}{N}, m = 0, 2 \ldots N-1$$

- Applies to discrete time and frequency signals.
- Same form of DFS but for aperiodic signals:
  signal treated as periodic for computational purpose only.

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ spectrum has period N

DFT properties

	Time	Frequency
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-j\frac{2\pi k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi h t}{T}} \cdot s[n]$	$S(k - h)$

⋮

Hilbert Spaces, Orthonormal Systems, and generalizations of the Fourier Transform

see blackboard