Computational Diagnostics & Biocybernetics Unit

# BMT 801 :
# Biomedizinische Signal- und Bildverarbeitung
# EF1
# („adhoc quick & dirty Einführung“)

Prof. Dr. Dr. Daniel J. Strauss



- Signalkategorien

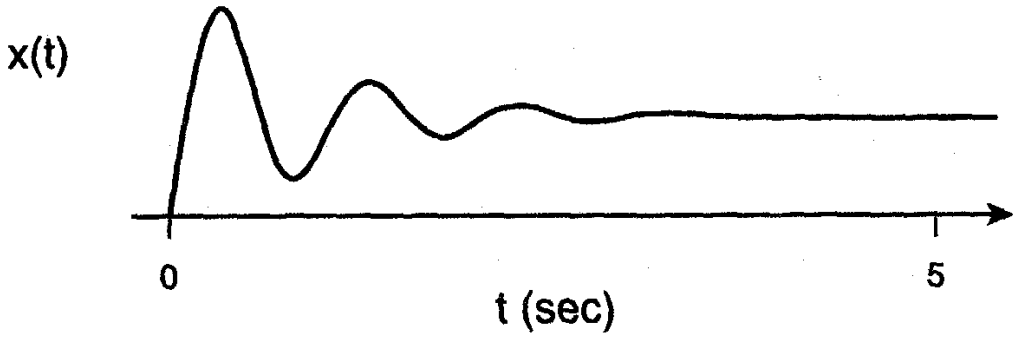
  - Amplituden- und Zeitkontinuierlich.

  - Amplitudenkontinuierlich und Zeitdiskret.

  - Amplitudendiskret und Zeitkontinuierlich.

  - Amplituden- und Zeitdiskret

# Amplituden- und Zeitkontinuierlich.

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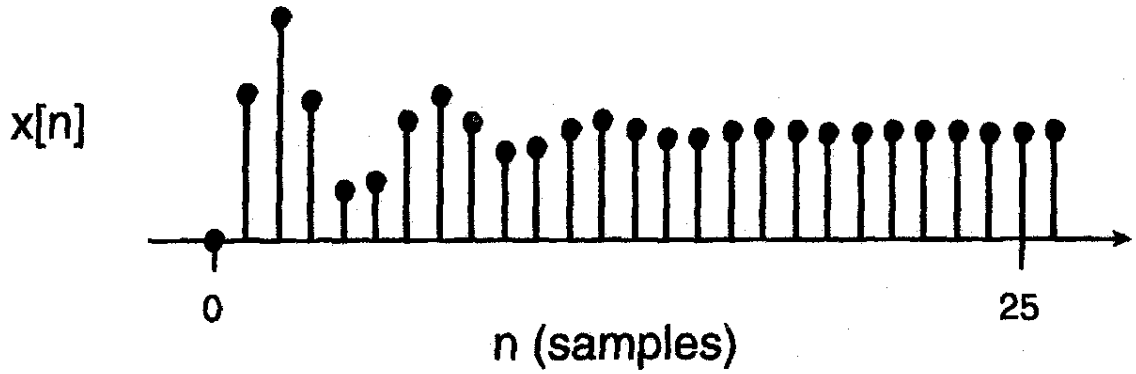
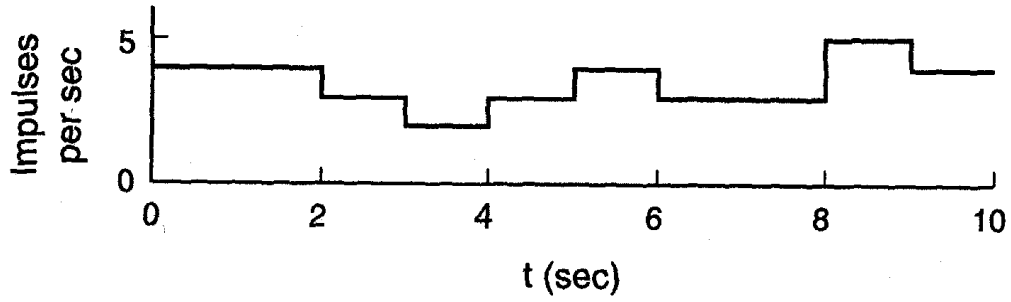
# Amplitudenkontinuierlich und Zeitdiskret.



# Amplitudendiskret und Zeitkontinuierlich.

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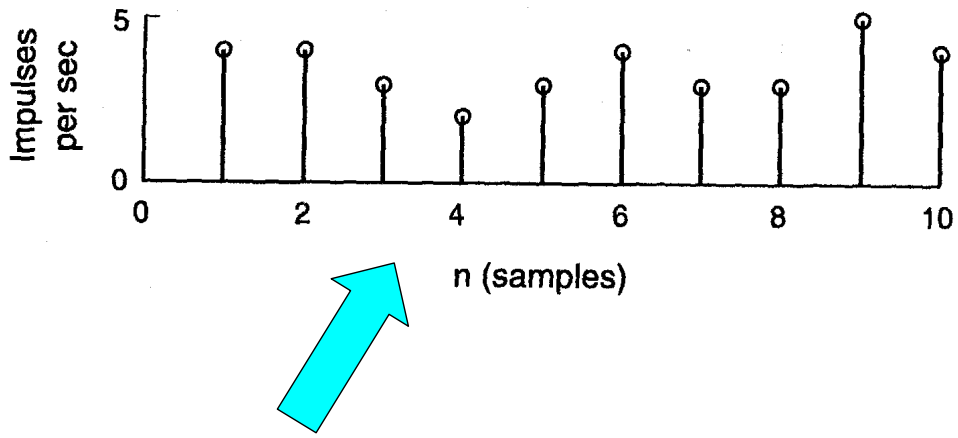
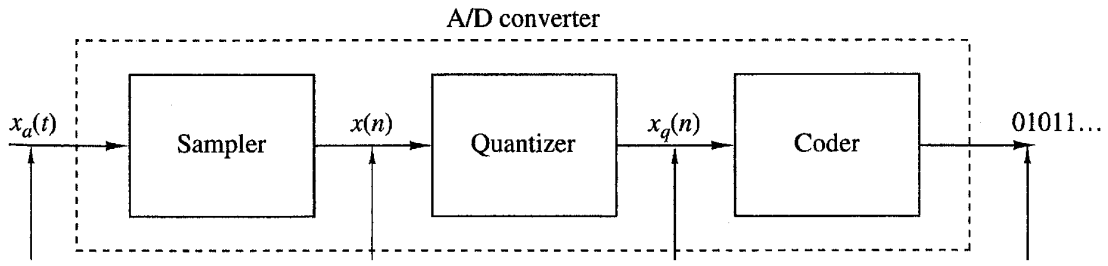
# Amplituden- und Zeitdiskret




Dies wird unser Fokus in der Vorlesung sein, warum?

# Analog- und Digitalwandler A/D-Konverter




Amplituden- und Zeitkontinuierlich

Amplitudenkontinuierlich und Zeitdiskret

Amplituden- und Zeitdiskret

Codiertes Signal 01100010011

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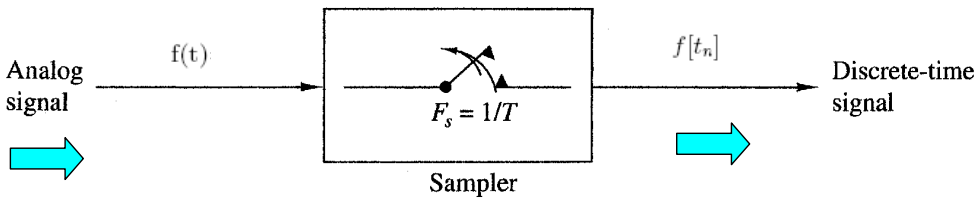
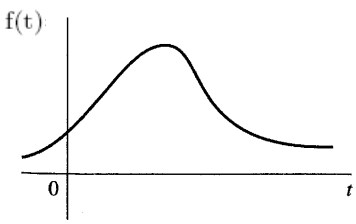
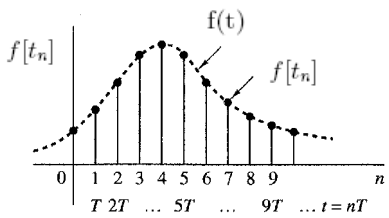
# Sampler

- einheitliche Abtastung mit konstanter Frequenz
- das Öffnen und Schließen des „Schalters“ erfolgt mit der Abtastfrequenz fs

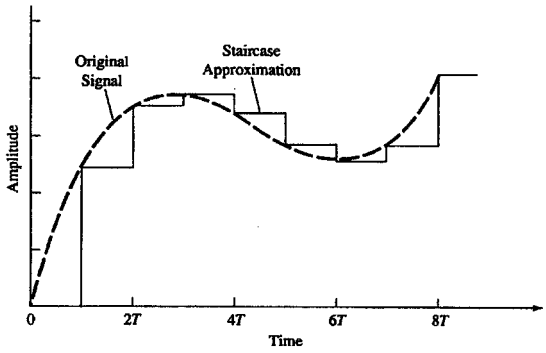


# Quantisierung

- Approximiert kontinuierliches Signal in eine endliche Anzahl von „äquidistanten Intervallen“.
- Intervall hat konstanten Wert
- Kleinster Schritt: Messbereich/Anzahl der Intervalle.

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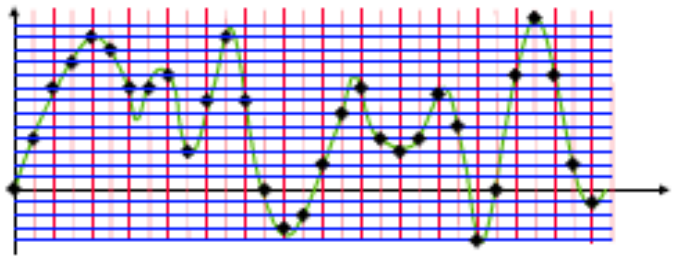
Kap. 1.2 Analog- und Digitalwandler

# Sampling und Quantisierung



**Beispiel: USBamp**

Messbereich $U_{ss} = \pm 250\ mV$

Auflösung $24\ Bit = 2^{24} = 16,8\ 10^6$

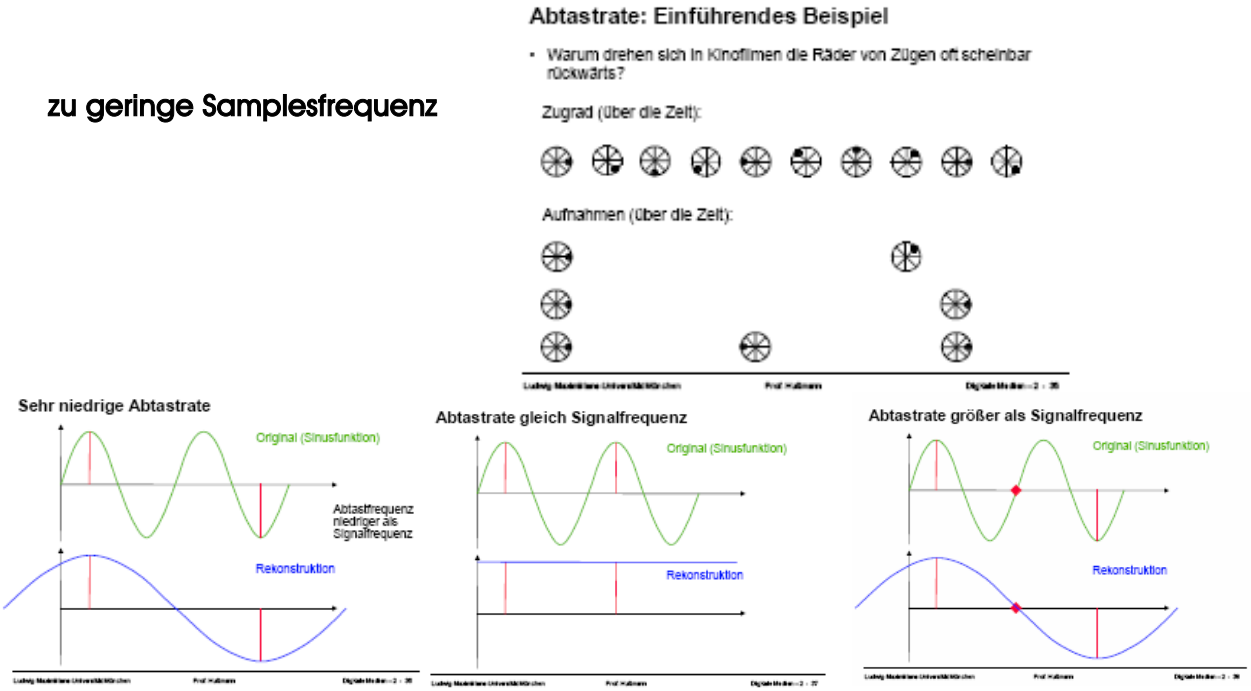
kleinstes Quantil $Q = \frac{U_{ss}}{N-1} = \frac{0,5\ V}{2^{24}-1} \approx 30\ nV$

Abtastfrequenz von 16 Hz bis 38 kHz

# Sampling und Quantisierung

## Fehlerquellen

zu geringe Samplesfrequenz

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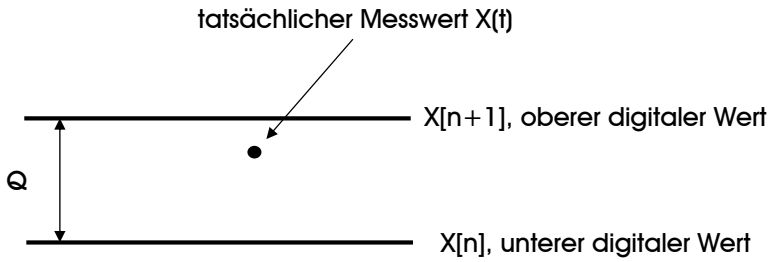
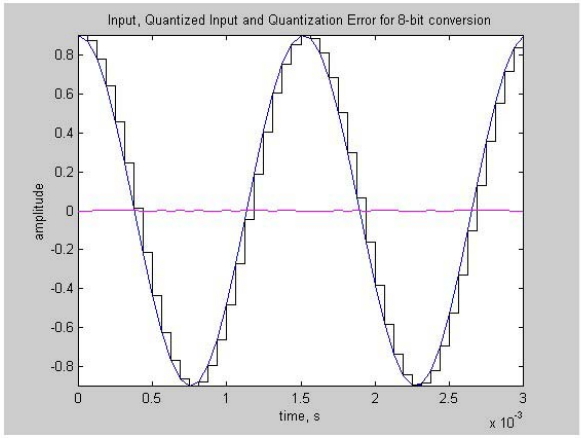
# Sampling und Quantisierung

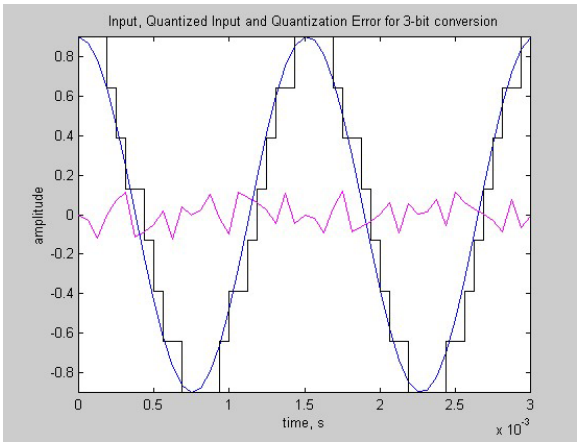
## Fehlerquellen

## Quantisierungsfehler




- minimaler Fehler 0, maximaler Fehler Q.
- mittlerer Fehler Q / 2.
- Fehler der Quantisierung wird auch „digitales Rauschen“ genannt.

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# Abtasttheorem

Das Spektrum einer Folge X[n] ergibt sich zu:

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

mit

$$e^{-j\omega t_0} = \mathcal{F}\left(\delta(t - t_0)\right)$$

erhält man

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, \mathcal{F}\left(\delta[t - nT]\right)$$

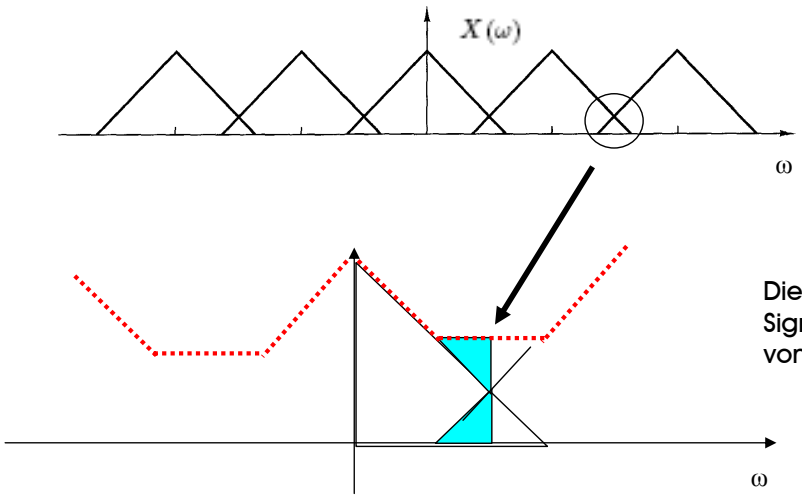
das Ergebnis

$$X(\omega) = \omega_T \cdot \sum_{n=-\infty}^{+\infty} X(\omega - \omega_T)$$

D.h. das Spektrum X(ω) der Folge X[n] ergibt sich aus der Überlagerung der jeweils um ein ganzzahliges Vielfaches der Periode 1/T verschobenen Spektren des analogen Signals.

# Abtasttheorem

Ein bandbegrenztes digitales Signal, welches mit einer zu geringen Abtastrate gemessen wurde, hat im Frequenzbereich eine Überlappung der Spektren.




Diese Überlappung führt bei der Signalsynthese zu einem Verlust von Informationen.

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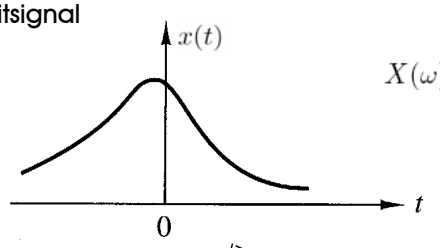
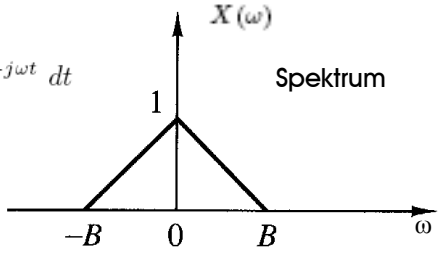
# Abtasttheorem

Zeitsignal

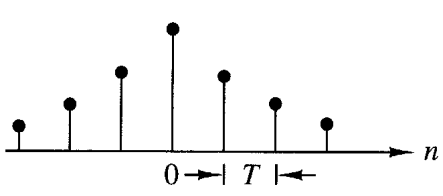


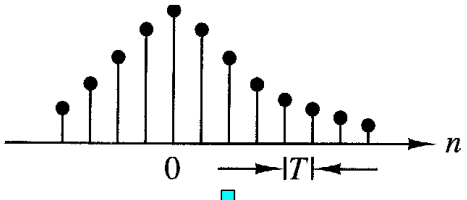

$$X(\omega) = \int_{-\omega}^{+\omega} x(t)\, e^{-j\omega t}\, dt$$




geringe Abtastrate

hohe Abtastrate

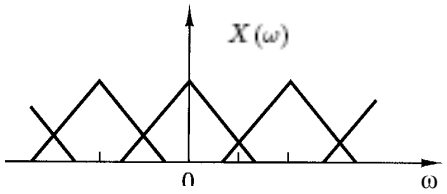


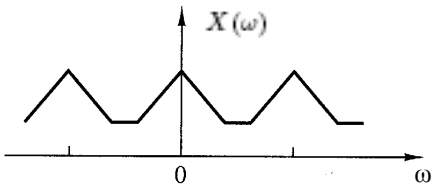



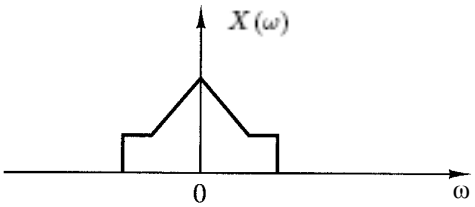

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$

# Abtasttheorem

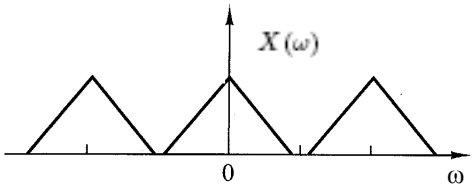
geringe Abtastrate




hohe Abtastrate




$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

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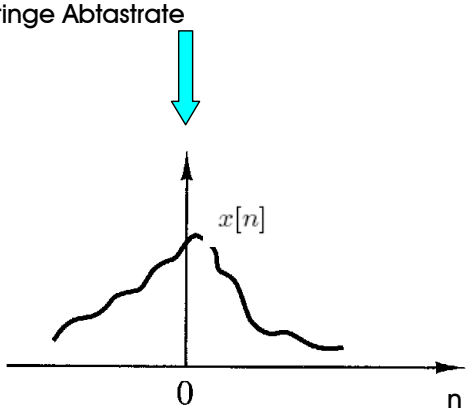
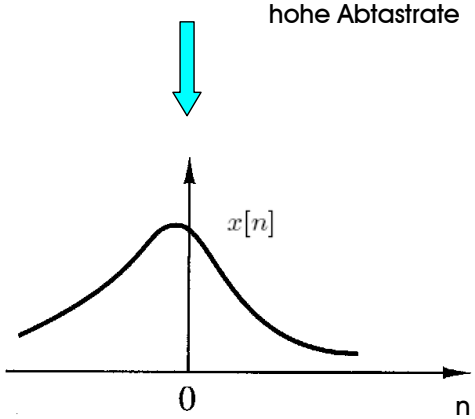
# Abtasttheorem

geringe Abtastrate




hohe Abtastrate




Die Abtastung im Zeitbereich führt zu einer Periodisierung im Frequenzbereich (und umgekehrt). Eine Überlappung der Frequenzspektren kann dadurch verhindert werden, dass das bandbegrenzte Eingangssignal mit einer deutlich über der Grenzfrequenz befindlichen Samplefrequenz abgetastet wird.

Nach Shannon muss die Samplefrequenz min. doppelt so hoch sein, wie die Grenzfrequenz des Eingangssignals.

$$f_s > 2 \cdot f_g$$

- Signaltypen
  - Deterministische Signale
  - Stochastische Signale
  - Chaotische Signale
  - Fraktale Signale

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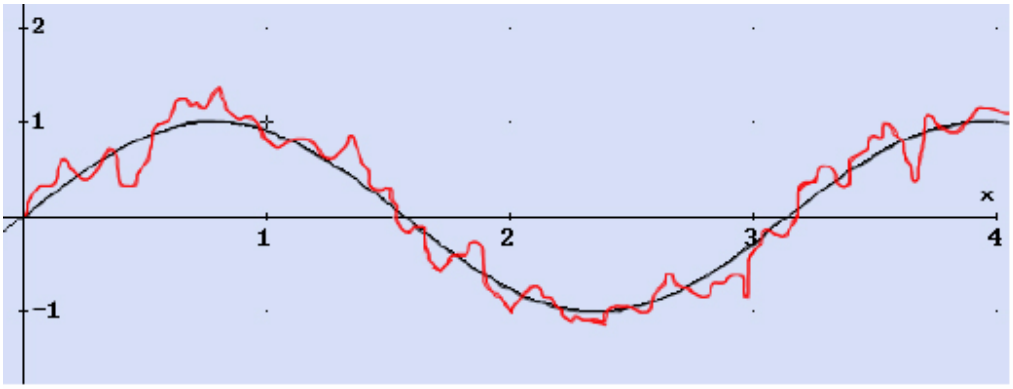
$$x(t) = a(t) + s(t)$$

x(t) = gemessenes Signal
a(t) = deterministischer Anteil
s(t) = Rauschanteil/stochastischer Signalanteil/Störsignal



# FOURIER ANALYSIS

*PART 1:*

*Frequency Analysis & Fourier Series*

**Slides are partly prepared by Maria Elena Angoletta**

**CERN (European Organization for Nuclear Research), Switzerland**

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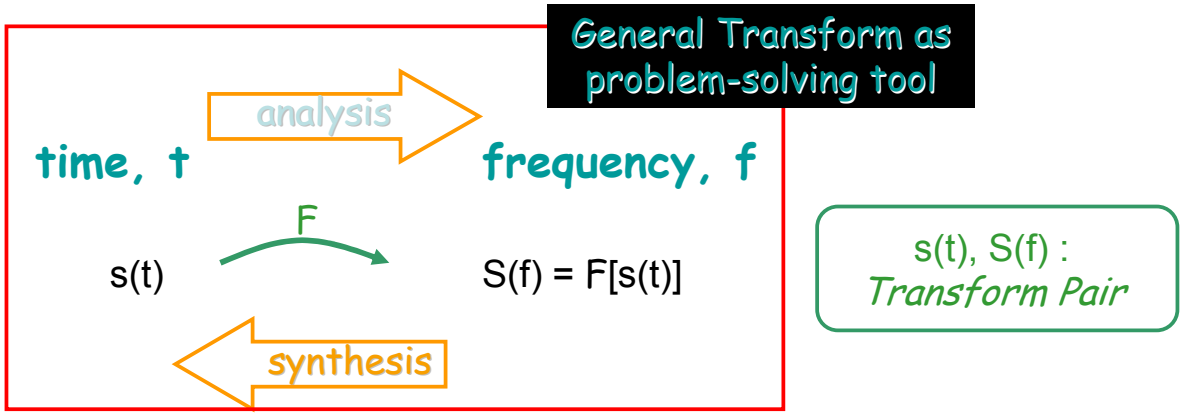
## TOPICS

1. Frequency analysis: a powerful tool
2. A tour of Fourier Transforms
3. Continuous Fourier Series (FS)
4. Discrete Fourier Series (DFS)

## Frequency analysis: why?

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns. (PDE).
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.

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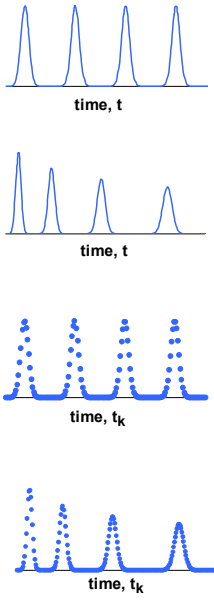
## Fourier analysis - applications

### Applications wide ranging and ever present in modern life

- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated "devices", biometry, …).

# Fourier analysis - tools




| Input Time Signal | | | Frequency spectrum | |
|---|---|---|---|---|
| Continuous | Periodic *(period T)* | FS | Discrete | $c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$ |
| | Aperiodic | FT | Continuous | $S(f) = \int_{-\infty}^{+\infty} s(t)\cdot e^{-j2\pi f t}dt$ |
| Discrete | Periodic *(period T)* | DFS** | Discrete | $\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$ |
| | Aperiodic | DTFT | Continuous | $S(f) = \sum_{n=-\infty}^{+\infty} s[n]\cdot e^{-j2\pi f n}$ |
| | | DFT** | Discrete | $\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$ |

*Note: $j=\sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples* ** *Calculated via FFT*

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# A little history

- Astronomic predictions by Babylonians/Egyptians likely via trigonometric sums.
- **1669**: Newton stumbles upon light spectra (*specter* = ghost) but fails to recognise "frequency" concept (*corpuscular* theory of light, & no waves).
- **18th century**: two outstanding problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation). **BUT** peers' consensus is that sum of sinusoids *only* represents smooth curves. Big blow to utility of such sums for all but Fourier ...
- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines. Strong criticism by peers blocks publication. Work published, 1822 *("Theorie Analytique de la chaleur")*.

# A little history -2

➢ **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

## CONTINUOUS

→ Fourier extends the analysis to arbitrary function (Fourier Transform).

→ Dirichlet, Poisson, Riemann, Lebesgue address FS convergence.

→ Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

## DISCRETE: Fast calculation methods (FFT)

→ **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).

→ **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).

→ Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).

→ FFT algorithm refined & modified for most computer platforms.

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# Fourier Series (FS)

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

**synthesis**

$$s(t) = a_0 + \sum_{k=1}^{+\infty} [a_k \cdot \cos(k\omega t) - b_k \cdot \sin(k\omega t)]$$

*For all t but discontinuities*

**analysis**

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)dt$$

*(signal average over a period, i.e. DC term & zero-frequency component.)*

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\omega t)dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\omega t)dt$$

**$a_0$, $a_k$, $b_k$** : Fourier coefficients.

**k**: harmonic number,

**T**: period, $\omega = 2\pi/T$

* see next slide

# FS convergence

## Dirichlet conditions

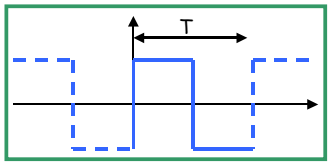
In any period:

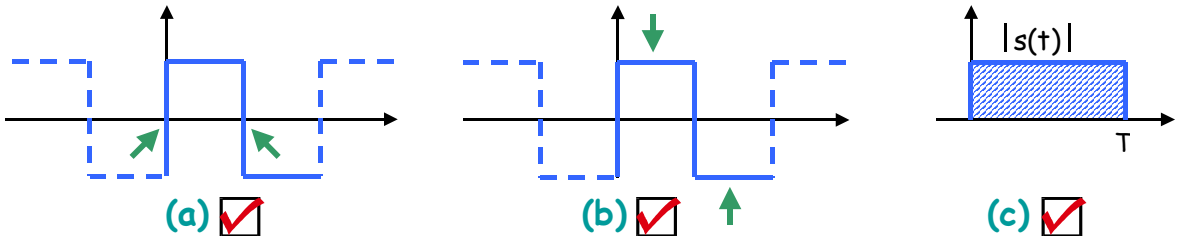
(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

(c) s(t) absolutely integrable, $\int_0^T |s(t)|\,dt < \infty$

Example: square wave




$x \le y$, then $f(x) \le f(y)$



# FS analysis - 1

FS of odd* function: square wave.

$T = 2\pi \Rightarrow \omega = 1$

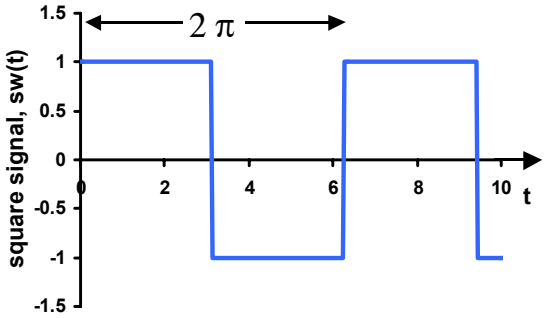
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\cos kt\,dt - \int_{\pi}^{2\pi}\cos kt\,dt\right\} = 0$$ *(odd function)*

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\sin kt\,dt - \int_{\pi}^{2\pi}\sin kt\,dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot\{1-\cos k\pi\} =$$

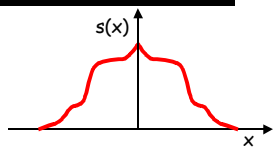
$$= \begin{cases} \frac{4}{k\cdot\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$

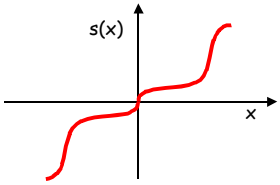



### * Even & Odd functions

**Even :** $s(-x) = s(x)$



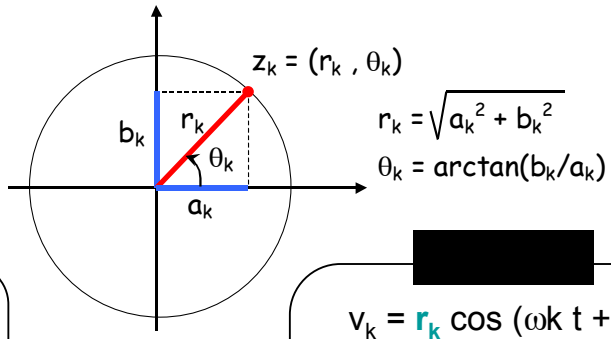

**Odd :** $s(-x) = -s(x)$



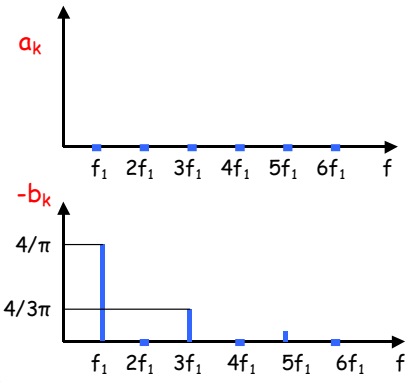
# FS analysis - 2

## Fourier spectrum representations

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$




$v_k = a_k \cos(\omega k\, t) - b_k \sin(\omega k\, t)$

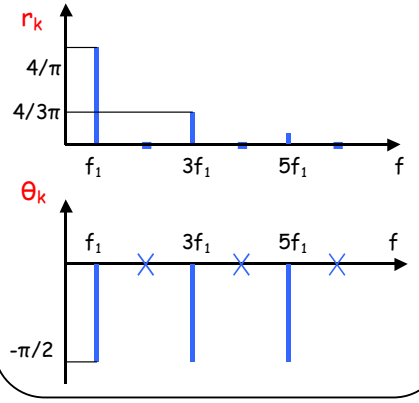



$r_K$
$\theta_K$

$f_k = k\,\omega/2\pi$

**Fourier spectrum of square-wave.**

$v_k = r_k \cos(\omega k\, t + \theta_k)$

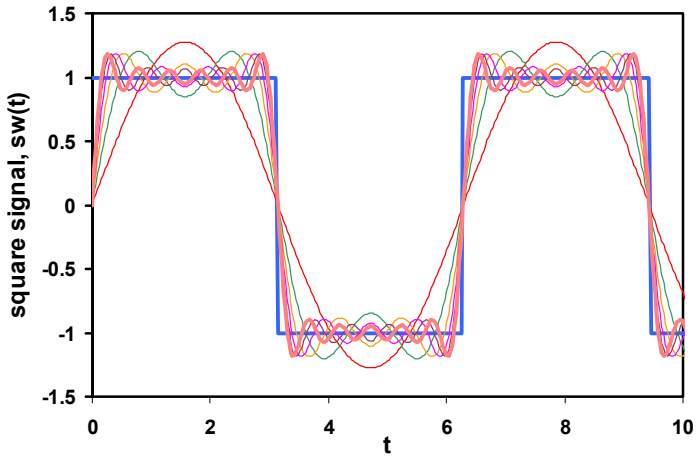





# FS synthesis

## Square wave reconstruction from spectral terms

$$sw(t) = \sum_{k=1}^{n} [b_k \sin(k\omega t)]$$

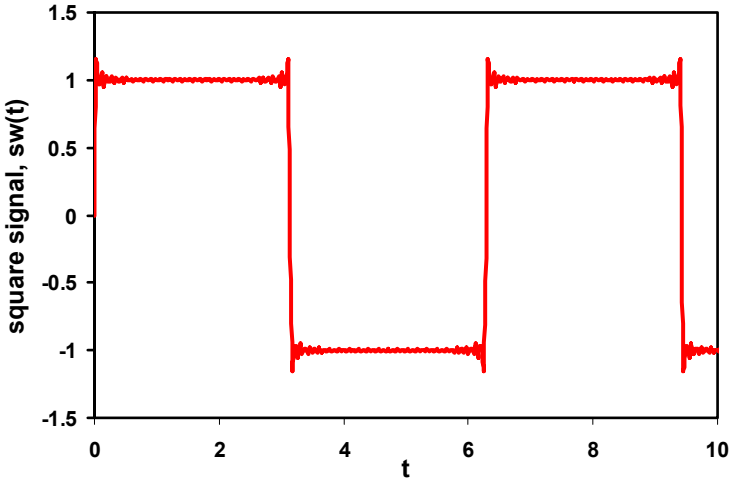



Convergence may be slow (~1/k) - ideally need infinite terms.
***Practically***, series truncated when remainder below computer tolerance (⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

# Gibbs phenomenon

## Overshoot exist @ each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} [-b_k \cdot \sin(kt)]$$




- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude. Just a minor annoyance.

Computational Diagnostics & Biocybernetics Unit

# FS time shifting

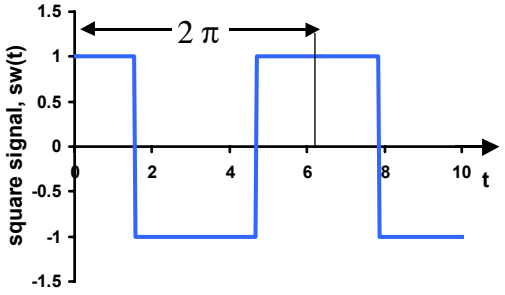
## FS of even function: π/2-advanced square-wave

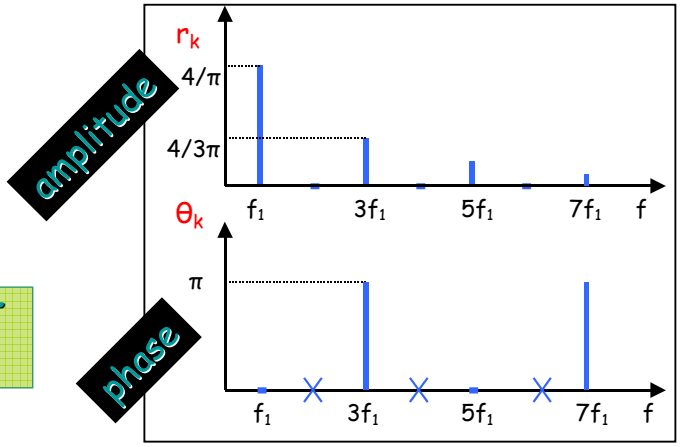
$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \frac{4}{k \cdot \pi}, & k \text{ odd}, \quad k = 1, 5, 9... \\ -\frac{4}{k \cdot \pi}, & k \text{ odd}, \quad k = 3, 7, 11... \\ 0, & k \text{ even}. \end{cases}$$

$-b_k = 0$ *(even function)*

Note: amplitudes unchanged ***BUT*** phases advance by k·π/2.



# Complex FS

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - j\cdot\sin(t)$ ⇒ "phasor"

$\cos(t) = \frac{e^{jt} + e^{-jt}}{2}$ $\sin(t) = \frac{e^{jt} - e^{-jt}}{2\cdot j}$

**analysis**

$$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$$

**synthesis**

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

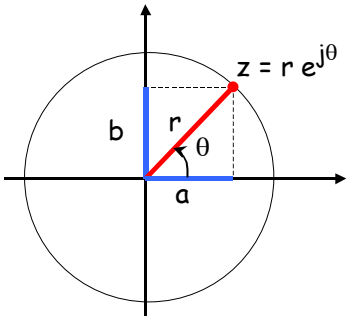
Complex form of FS (Laplace 1782). Harmonics $c_k$ separated by $\Delta f = 1/T$ on frequency plot.

***Note***: $c_{-k} = (c_k)^*$

**Link to FS real coeffs.**

$c_0 = a_0$

$$c_k = \frac{1}{2}\cdot(a_k + j\cdot b_k) = \frac{1}{2}\cdot(a_{-k} - j\cdot b_{-k})$$




$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$



# FS properties

| | Time | Frequency |
|---|---|---|
| Homogeneity | a·s(t) | a·S(k) |
| Additivity | s(t) + u(t) | S(k)+U(k) |
| Linearity | a·s(t) + b·u(t) | a·S(k)+b·U(k) |
| Time reversal | s(-t) | S(-k) |
| Multiplication * | s(t)·u(t) | $\sum_{m=-\infty}^{\infty} S(k-m)U(m)$ |
| Convolution * | $\frac{1}{T}\cdot\int_0^T s(t-\bar{t})\cdot u(\bar{t})d\bar{t}$ | S(k)·U(k) |
| Time shifting | $s(t-\bar{t})$ | $e^{-j\frac{2\pi k\cdot\bar{t}}{T}}\cdot S(k)$ |
| Frequency shifting | $e^{+j\frac{2\pi m t}{T}}\cdot s(t)$ | S(k - m) |

# FS - power

*Average power W* : $W = \frac{1}{T}\int_{o}^{T} |s(t)|^2 \, dt \equiv s(t) \otimes s(t)$

**Parseval's Theorem**

$$W = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)$$

- FS convergence ~1/k

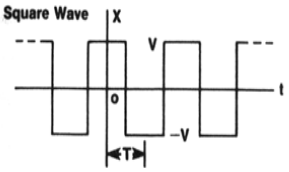
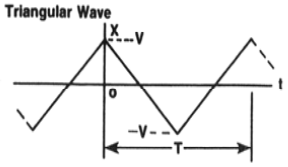
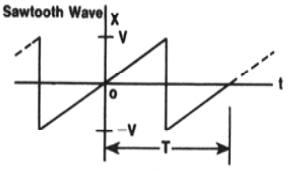
  ⇒ *lower frequency terms*

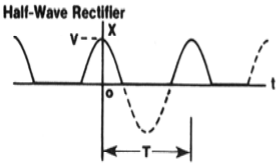
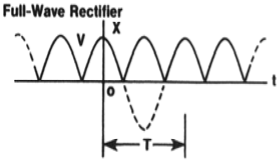
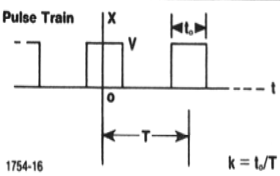
  $W_k = |c_k|^2$ *carry most power.*

- $W_k$ vs. $\omega_k$: Power density spectrum.



# FS of main waveforms

| Wave Shape | Fourier Series -- $\omega_o = 2\pi/T$ |
|---|---|
| Square Wave  | $x(t)=\frac{4V}{\pi}(\cos\omega_o t-\frac{1}{3}\cos 3\omega_o t+\frac{1}{5}\cos 5\omega_o t-\frac{1}{7}\cos 7\omega_o t+\ldots)$ |
| Triangular Wave  | $x(t)=\frac{8V}{\pi^2}(\cos\omega_o t+\frac{1}{9}\cos 3\omega_o t+\frac{1}{25}\cos 5\omega_o t+\ldots)$ |
| Sawtooth Wave  | $x(t)=\frac{2V}{\pi}(\sin\omega_o t-\frac{1}{2}\sin 2\omega_o t+\frac{1}{3}\sin 3\omega_o t-\frac{1}{4}\sin 4\omega_o t+\ldots)$ |

| Wave Shape | Fourier Series -- $\omega_o = 2\pi/T$ |
|---|---|
| Half-Wave Rectifier  | $x(t)=\frac{V}{\pi}(1+\frac{\pi}{2}\cos\omega_o t+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\ldots\ldots(-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\ldots)$ n even |
| Full-Wave Rectifier  | $x(t)=\frac{2V}{\pi}(1+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\ldots\ldots(-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\ldots)$ n even |
| Pulse Train  1754-16 | $x(t)=V[k+\frac{2}{\pi}(\sin k\pi\cos\omega_o t+\frac{1}{2}\sin 2k\pi\cos 2\omega_o t+\ldots\ldots+\frac{1}{n}\sin nk\pi\cos n\omega_o t+\ldots)]$ $k=t_o/T$ |

# Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

**DFS defined as:**

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

***Note:*** $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ same period N

*i.e. time periodicity propagates to frequencies!*

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Synthesis: finite sum $\Leftarrow$ band-limited s[n]

DFS generate periodic $c_k$ with same signal period

**Orthogonality in DFS**

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

Kronecker's delta

N consecutive samples of s[n] completely describe s in time or frequency domains.

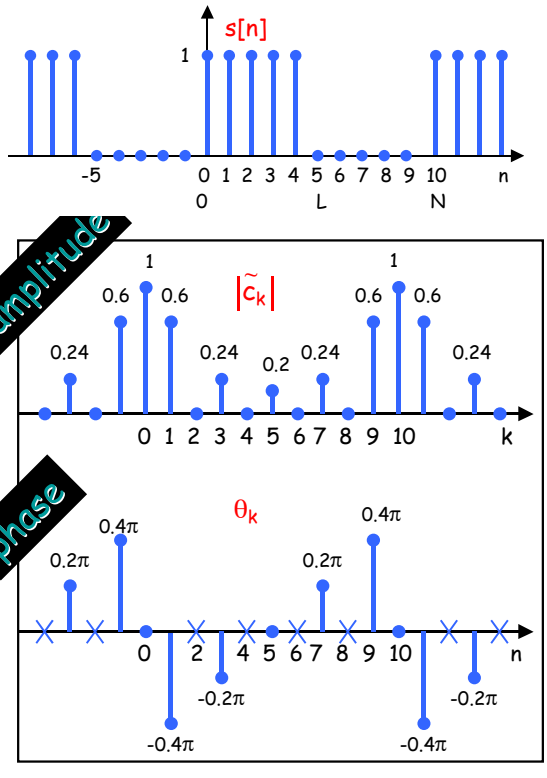


# DFS analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \frac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\ \frac{e^{-j\frac{\pi k(L-1)}{N}}}{N}\cdot\frac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$




*Discrete signals* $\Rightarrow$ *periodic frequency spectra.*
*Compare to continuous rectangular function*

# DFS properties

| | Time | Frequency |
|---|---|---|
| Homogeneity | $a \cdot s[n]$ | $a \cdot S(k)$ |
| Additivity | $s[n] + u[n]$ | $S(k)+U(k)$ |
| Linearity | $a \cdot s[n] + b \cdot u[n]$ | $a \cdot S(k)+b \cdot U(k)$ |
| Multiplication * | $s[n] \cdot u[n]$ | $\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$ |
| Convolution * | $\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$ | $S(k) \cdot U(k)$ |
| Time shifting | $s[n - m]$ | $e^{-j\frac{2\pi k \cdot m}{T}} \cdot S(k)$ |
| Frequency shifting | $e^{+j\frac{2\pi h t}{T}} \cdot s[n]$ | $S(k - h)$ |

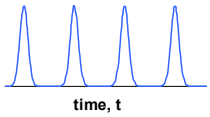
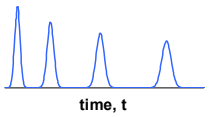
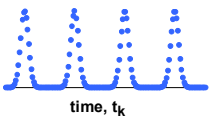
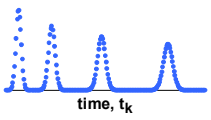
* *Explained in next week's lecture*

Computational Diagnostics & Biocybernetics Unit

# TOPICS

1. Infinite Fourier Transform (FT)
2. FT & generalised impulse
3. Uncertainty principle
4. Discrete Time Fourier Transform (DTFT)
5. Discrete Fourier Transform (DFT)
6. Comparing signal by DFS, DTFT & DFT

# Fourier analysis - tools

| Input Time Signal | | | Frequency spectrum | |
|---|---|---|---|---|
|  time, t | Continuous | Periodic *(period T)* — FS | Discrete | $c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}\,dt$ |
|  time, t | | Aperiodic — FT | Continuous | $S(f)=\int_{-\infty}^{+\infty} s(t)\cdot e^{-j2\pi f t}\,dt$ |
|  time, $t_k$ | Discrete | Periodic *(period T)* — DFS | Discrete | $\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi kn}{N}}$ |
|  time, $t_k$ | | Aperiodic — DTFT | Continuous | $S(f)=\sum_{n=-\infty}^{+\infty} s[n]\cdot e^{-j2\pi fn}$ |
| | | Aperiodic — DFT | Discrete | $\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi kn}{N}}$ |

*Note: $j=\sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples*

Computational Diagnostics & Biocybernetics Unit

# Fourier Integral (FI)

## Fourier analysis tools for aperiodic signals.

**(1)** s(t) continuous, s'(t) monotonic

**(2)** $\int_{-\infty}^{+\infty} |s(t)|\,dt < +\infty$

**Fourier Integral Theorem**

*Any aperiodic signal s(t) can be expressed as a Fourier integral if s(t) piecewise smooth(1) in any finite interval (-L,L) and absolute integrable(2).*

$$s(t)=\int_0^{+\infty}\{A(\omega)\cdot\cos(\omega t)+B(\omega)\cdot\sin(\omega t)\}d\omega \quad (3)$$

**(3)** $A(\omega)=\frac{1}{\pi}\cdot\int_{-\infty}^{+\infty} s(t)\cos(\omega t)\,dt \qquad B(\omega)=\frac{1}{\pi}\cdot\int_{-\infty}^{+\infty} s(t)\sin(\omega t)\,dt$

*Real-to-complex link*

$$S(\omega)=\pi\cdot[A(\omega)-j\cdot B(\omega)]$$

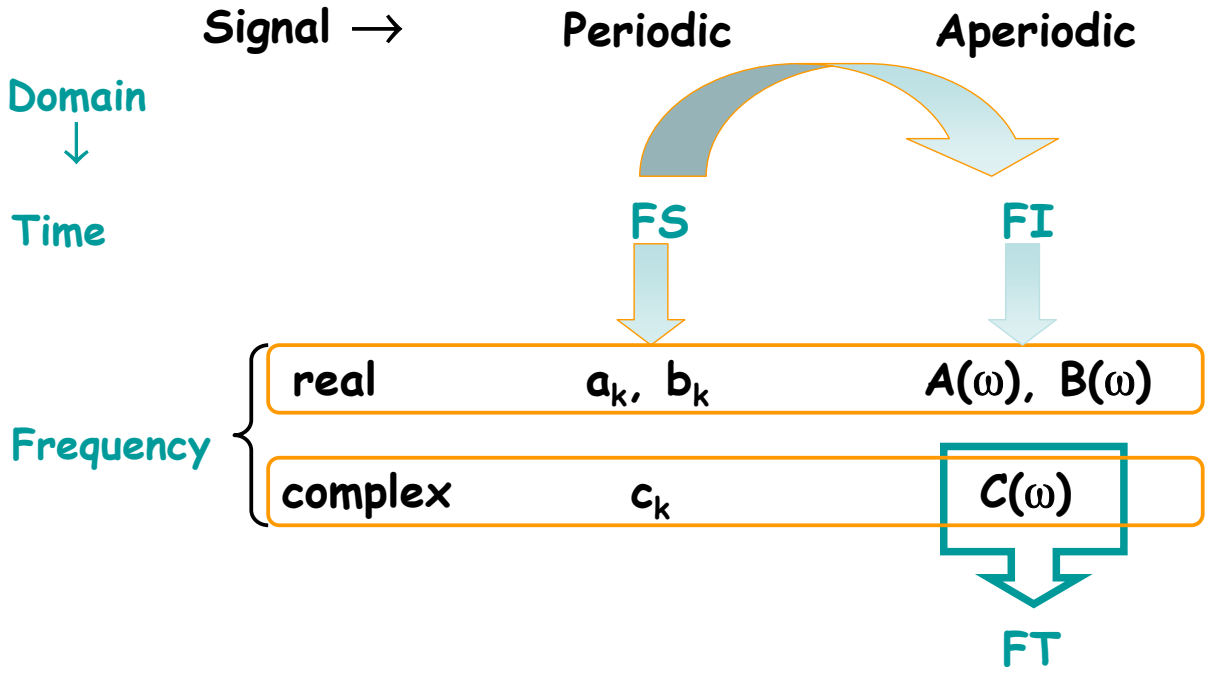
*Complex form*

analysis: $S(\omega)=\int_{-\infty}^{+\infty} s(t)\cdot e^{-j\omega t}\,dt$

synthesis: $s(t)=\frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} S(\omega)\cdot e^{j\omega t}\,d\omega$

Fourier Transform (Pair) - FT

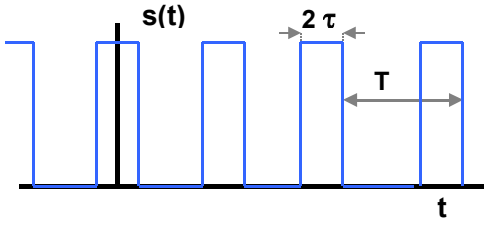
# Let's summarise a little



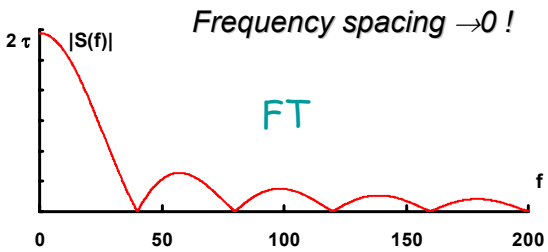


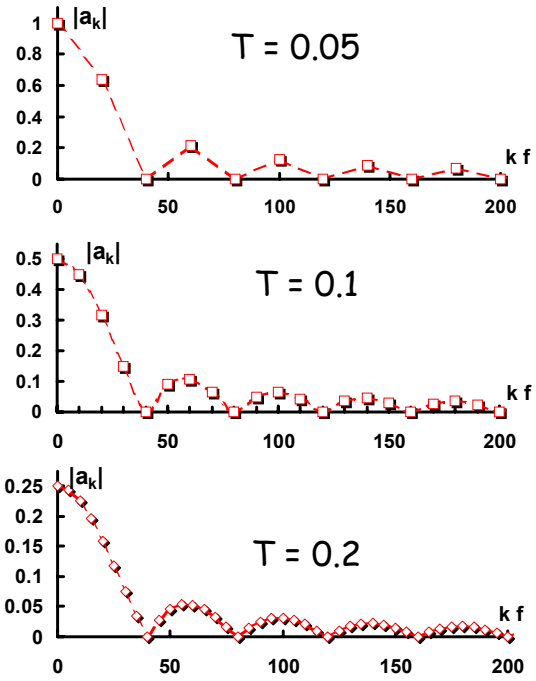

# From FS to FT

FS moves to FT as period T increases: ➡ continuous spectrum




Pulse train, width 2 τ = 0.025




*Note: $|a_k| \to 2\,a_0$ as $k \to 0 \Rightarrow 2\,a_0$ is plotted at k=0*

# Getting FS from FT

**FS defined**

$$c_k = \frac{1}{T}\cdot\int_{-T/2}^{T/2} s(t)\cdot e^{-jk\omega t}\,dt$$

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

$\Delta f = \Delta\omega/(2\pi) = 1/T$
**frequency spacing**

**2**

$$\Gamma_{\omega_k} \equiv \frac{c_{\omega_k}}{\Delta f} = \int_{-T/2}^{T/2} s(t)\cdot e^{-j\,\omega_k\,t}dt$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} \Gamma_{\omega_k}\cdot e^{j\,\omega_k\,t}\,\Delta f$$

As $\Delta f \to 0$ , replace $\Delta f$ , $\omega_K$ , $\sum_{\omega_k=-\infty}^{\infty}$ by df, $2\pi f$, $\int_{-\infty}^{+\infty}$

**1**

$$c_k = \Delta f\cdot\int_{-T/2}^{T/2} s(t)\cdot e^{-j\,\omega_k\,t}dt \equiv c_{\omega_k}$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} c_{\omega_k}\cdot e^{j\,\omega_k\,t}$$

**FT defined**

$$S(f) = \lim_{\Delta f\to 0}\Gamma_{\omega_k} = \int_{-\infty}^{+\infty} s(t)\cdot e^{-j\omega t}\,dt$$

$$s(t) = \int_{-\infty}^{+\infty} S(f)\cdot e^{j2\pi ft}\,df$$

Computational Diagnostics & Biocybernetics Unit

# FT properties

| | Time | Frequency |
|---|---|---|
| Linearity | $a\cdot s(t) + b\cdot u(t)$ | $a\cdot S(f)+b\cdot U(f)$ |
| Multiplication | $s(t)\cdot u(t)$ | $\int_{-\infty}^{+\infty} S(f-\bar{f})\,U(\bar{f})\,d\bar{f}$ |
| Convolution | $\int_{-\infty}^{+\infty} s(t-\bar{t})\cdot u(\bar{t})\,d\bar{t}$ | $S(f)\cdot U(f)$ |
| Time shifting | $s(t-\bar{t})$ | $e^{-j\,2\pi\,f\,\bar{t}}\cdot S(f)$ |
| Frequency shifting | $e^{+j2\pi\bar{f}}\cdot s(t)$ | $S(f-\bar{f})$ |
| Time reversal | $s(-t)$ | $S(-f)$ |
| Differentiation | $\frac{ds(t)}{dt}$ | $j2\pi f\,S(f)$ |
| Parseval's identity | $\int h(t)\,g^*(t)\,dt = \int H(f)\,G^*(f)\,df$ | |
| Integration | $\int_{-\infty}^{t} s(u)\,du$ | $S(f)/(j2\pi f)$ |
| Energy & Parseval's (E is t-to-f invariant) | $E = \int_{-\infty}^{+\infty} \lvert s(t)\rvert^2 dt = \int_{-\infty}^{+\infty} \lvert S(f)\rvert^2 df$ | |

# FT - uncertainty principle

**Bandwidth Theorem**

For effective duration $\Delta t$ & bandwidth $\Delta f$

$\exists\, \gamma > 0 \quad \Delta t \cdot \Delta f \geq \gamma$

uncertainty product

**Fourier uncertainty principle**

For Energy Signals:

$E = \int |s(t)|^2 dt = \int |S(f)|^2 df < \infty$

Define mean values

$$\bar{t}^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} t \cdot |s(t)|^2 \, dt \qquad \bar{f}^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} f \cdot |S(f)|^2 \, df$$

Define std. dev.

$$\Delta t^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} (t - \bar{t}) \cdot |s(t)|^2 dt \qquad \Delta f^2 = \frac{1}{E} \cdot \int_{-\infty}^{+\infty} (f - \bar{f}) \cdot |S(f)|^2 df$$

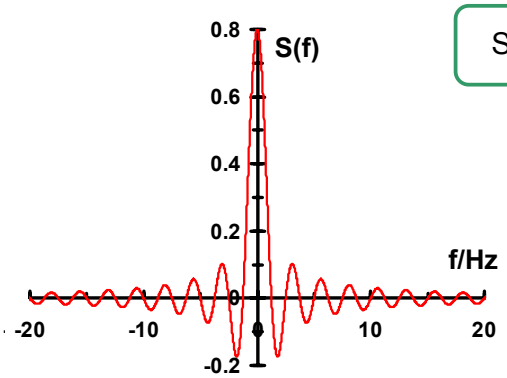
$\Rightarrow \Delta t \cdot \Delta f \geq 1/4\pi$

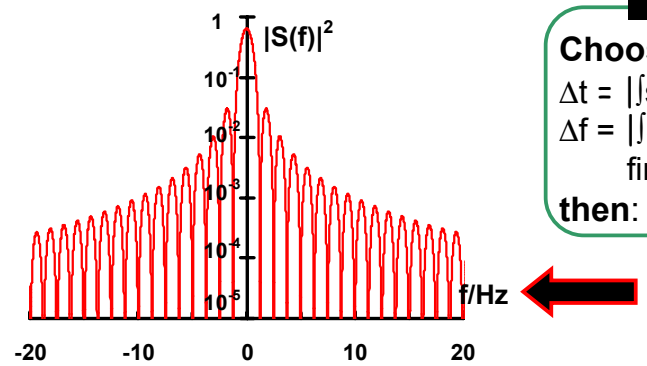
**Implications**

- Limited accuracy on simultaneous observation of s(t) & S(f).
- Good time resolution (small $\Delta t$) requires large bandwidth $\Delta f$ & vice-versa.

Computational Diagnostics & Biocybernetics Unit

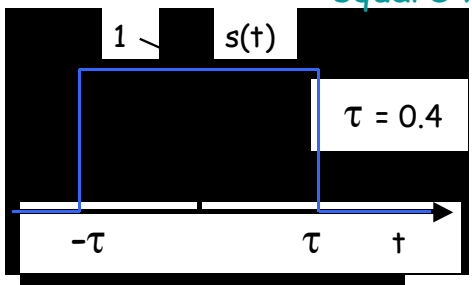
# FT - example




$S(f) = 2\tau\, s_{MAX}\, \mathrm{sync}(2f\tau)$

FT of $2\tau$-wide square window




Fourier uncertainty

**Choose**

$\Delta t = |\int s(t)/s(0)\, dt| = 2\tau$,

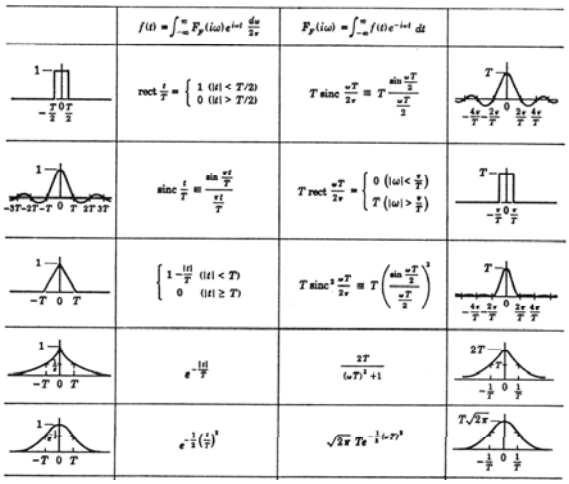
$\Delta f = |\int S(f)/S(0)\, df| = 1/(2\tau)$ = half distance btwn first 2 zeroes ($f_{1,-1} = \pm 1/2\tau$) of S(f)

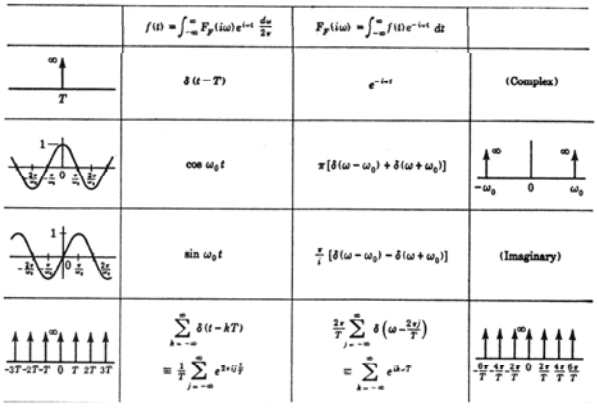
**then:** $\Delta t \cdot \Delta f = 1$

Power Spectral Density (PSD) vs. frequency f plot.

*Note:* Phases unimportant!

# FT of main waveforms





Computational Diagnostics & Biocybernetics Unit

# Discrete Time FT (DTFT)

**DTFT defined as:**

analysis

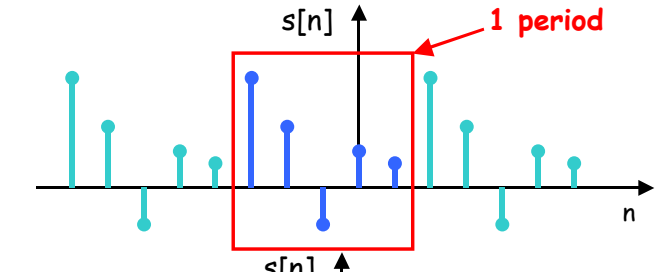
$$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi fn}$$

synthesis

$$s[n] = \frac{1}{2\pi} \cdot \int_{0}^{2\pi} S(f) e^{j2\pi fn} df$$

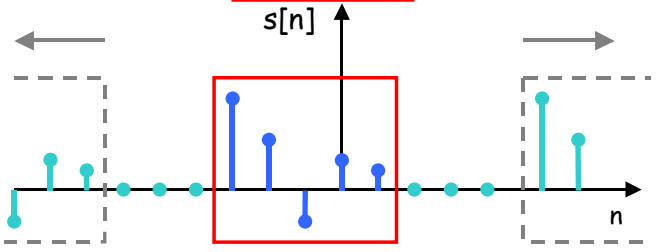
*Note*: continuous frequency domain! (frequency density function)

Holds for aperiodic signals



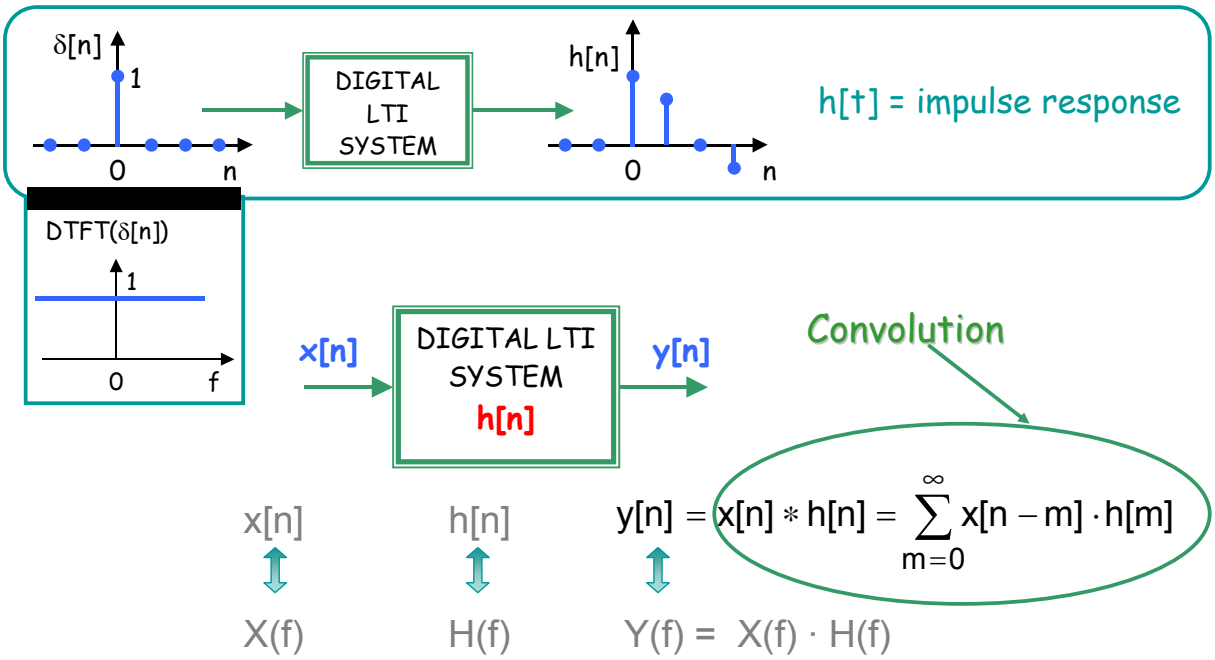

Obtained from DFS as $N \to \infty$

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi kn}{N}}$$



# DTFT - convolution

Digital Linear Time Invariant system: obeys superposition principle.




kernel of FT as eigenfunction of LTI systems – see blackboard



# Discrete FT (DFT)

**DFT defined as:**

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

**Frequency resolution**

**Analysis frequencies $f_m$**

$$f_m = \frac{m\cdot f_S}{N}, m = 0,2\ldots N-1$$

- Applies to discrete time and frequency signals.
- Same form of DFS but for aperiodic signals: signal treated as periodic for computational purpose only.

***Note:*** $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ spectrum has period N

# DFT properties

| | Time | Frequency |
|---|---|---|
| Linearity | $a \cdot s[n] + b \cdot u[n]$ | $a \cdot S(k) + b \cdot U(k)$ |
| Multiplication | $s[n] \cdot u[n]$ | $\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$ |
| Convolution | $\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$ | $S(k) \cdot U(k)$ |
| Time shifting | $s[n - m]$ | $e^{-j\frac{2\pi k \cdot m}{T}} \cdot S(k)$ |
| Frequency shifting | $e^{+j\frac{2\pi h t}{T}} \cdot s[n]$ | $S(k - h)$ |

Computational Diagnostics & Biocybernetics Unit

Hilbert Spaces, Orthonormal Systems, and generalizations of the Fourier Transform

see blackboard