Frequency Response (1)

- Transfer Function H(z) is the z-transform of the impulse response h(n)

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n}$$

- Output y(n) is the convolution of the input x(n) and impulse response h(n)

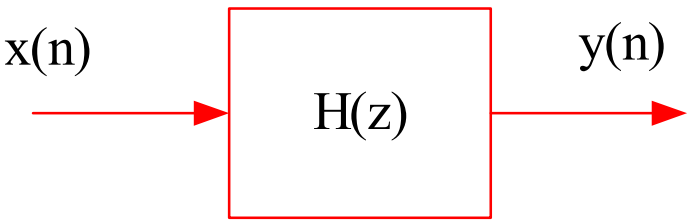
$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{+\infty} h(k)\, x(n-k)$$

Frequency Response (2)

- Consider a **pure phasor input** $x(n) = e^{j2\pi\Omega n}$
- The output y(n) is $y(n) = h(n) * x(n)$

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{+\infty} h(k)\, e^{j2\pi\Omega(n-k)}$$

$$y(n) = e^{j2\pi\Omega n} \left[\sum_{k=-\infty}^{+\infty} h(k)\, e^{-j2\pi\Omega k} \right]$$



$$\boxed{y(n) = e^{j2\pi\Omega n}\, H(e^{j2\pi\Omega})}$$

The response to a phasor is

✓ a phasor at the **same frequency**

✓ the gain is the **modulus** of $H(e^{j2\pi\Omega})$

✓ the phase shift is the **argument** of $H(e^{j2\pi\Omega})$

Difference Equation (1)

Linear Time-Invariant -> Constant Coefficient Linear Difference Equation

$$y(n) = a\, y(n-1) + x(n)$$

ICs: $y(-1) = 0$ then for $x(n) = \delta(n)$

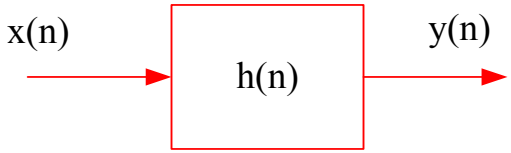
we get

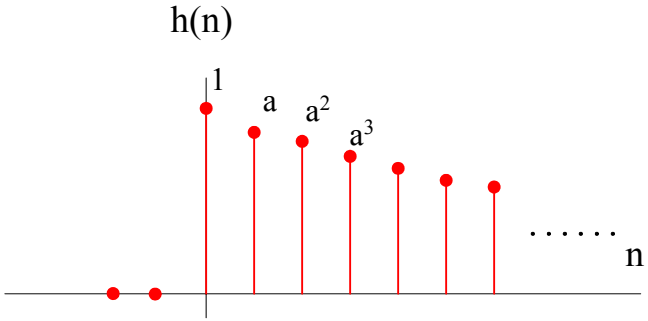
$y(0) = 1$

$y(1) = a\, y(0) + x(1) = a$

$y(2) = a\, y(1) + x(2) = a^2$

$$\rightarrow h(n) = a^n\, U(n)$$





Knowledge of underlying physics gives the difference equation.
Measurement gives impulse response.

Difference Equation (2)

Let y(n) be linear combination of the N past output values and the present and M past input values (causal) we get:

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \cdots + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots$$

or (with $a_0 = 1$)

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{r=0}^{M} b_r x(n-r)$$

General **difference equation** representing a **causal linear time-invariant filter.**

Rational Transfer Function

- Now take the z-transform of both sides

$$\sum_{k=0}^{N} a_k z^{-k} \quad Y(z) = \sum_{r=0}^{M} b_r z^{-r} \quad X(z)$$

- The filter transfer function H(z) is

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}}$$

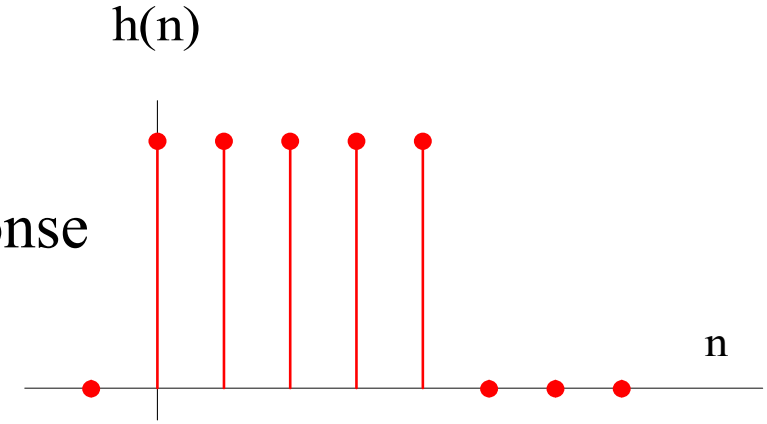
- H(z) is **rational**, with a Mth order polynomial as numerator and N^{th} order polynomial as denominator -> it has **M zeros and N poles in the z-plane**

FIR versus IIR (1)

- FIR = **Finite** Impulse Response filter

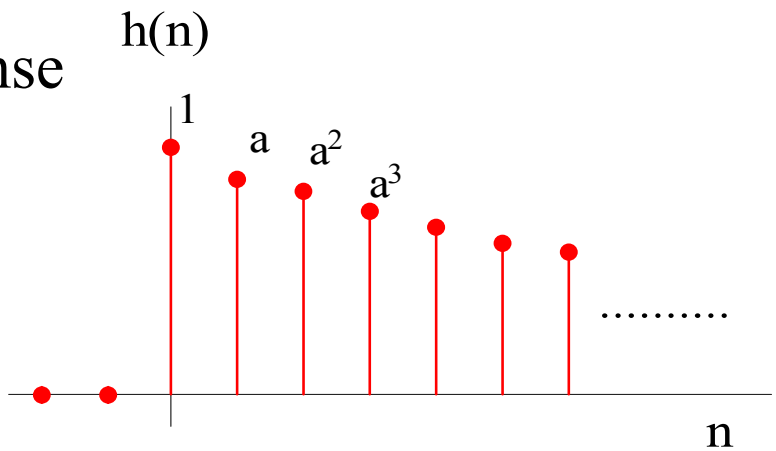
  h(n) non zero only for n in $[n_1,n_2]$

  example: rectangular response



- IIR = **Infinite** Impulse Response filter

  example: $h(n) = a^n U(n)$



FIR versus IIR (3)

- FIR transfer function

$$H(z) = \sum_{n=n_1}^{n_2} h(n) \quad z^{-n} = h(n_1)\, z^{-n_1} + h(n_1+1) z^{-n_1+1} + \cdots + h(n_2)\, z^{-n_2}$$

no pole except at z=0 (causal) or z=infinity (anti-causal)

- IIR transfer function

$$H(z) = \frac{\sum_{r=0}^{M} b_r\, z^{-r}}{\sum_{k=0}^{N} a_k\, z^{-k}} = \sum_{n=\ \infty}^{+\infty} h(n)\, z^{-n}$$

has **both poles and zeros** in the z-plane

FIR properties

- Frequency response is defined by location of **zeros only**. For a given complexity (nbr of multiply/add) the **transition band** will be **smoother** (that is less sharp) than with a IIR.
- The FIR are **always stable**:

$$y(n)=h(n)*x(n)=\sum_{k=-\infty}^{+\infty} h(k)\, x(n-k)=\sum_{k=n_1}^{n_2} h(k)\, x(n-k)$$

$$|y_{max}| \leqslant \left|\sum_{k=n_1}^{n_2} h(k)\, x(n-k)\right| \leqslant \left|\sum_{k=n_1}^{n_2} h(k)\right| \quad |x_{max}| \leqslant \left[\sum_{k=n_1}^{n_2} |h(k)|\right] \quad |x_{max}|$$

Even-symmetric FIR

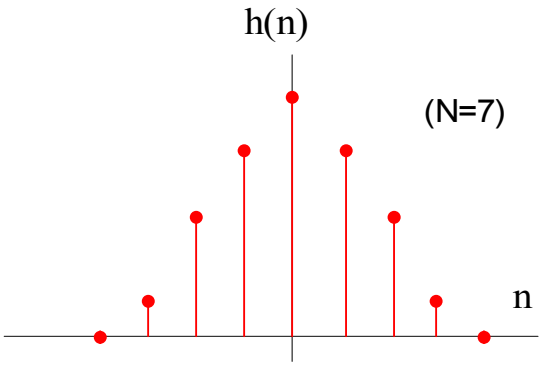
- $h(n) = h(-n)$
- Evaluate the frequency response (assuming that N is odd) and h(n) is real-valued

$$H(z) = \sum_{n=n_1}^{n_2} h(n) \quad z^{-n} = \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} h(n) \quad z^{-n}$$

if $h(n) = h(-n)$ we get

$$H(e^{j2\pi\Omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \quad (e^{-j2\pi n\Omega} - e^{+j2\pi n\Omega})$$

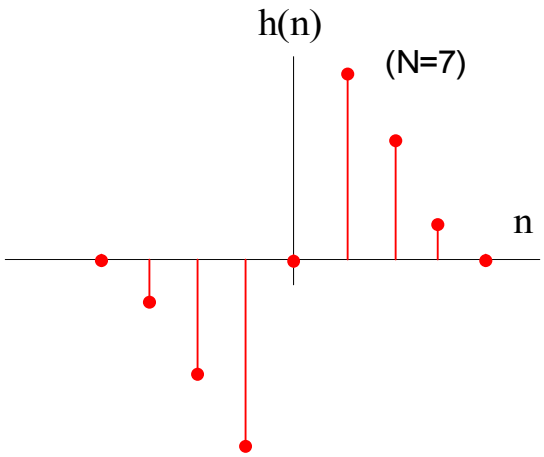
$$H(e^{j2\pi\Omega}) = h(0) + 2\sum_{n=1}^{\frac{N-1}{2}} h(n) \quad \cos[2\pi n\Omega]$$



The **frequency response is real**: phase shift is 0 or 180 degrees

Odd-symmetric FIR

- $h(n) = -h(-n)$
- Evaluate the frequency response (assuming that N is odd) and h(n) is real-valued



$$H(e^{j2\pi\Omega}) = -2j\sum_{n=1}^{\frac{N-1}{2}} h(n) \quad \sin[2\pi n\Omega]$$

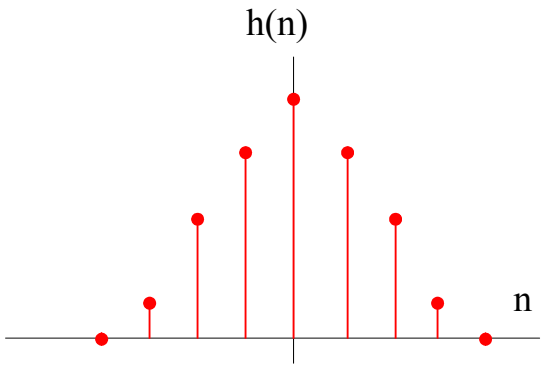
The **frequency response is imaginary**: phase shift is 90 or -90 degrees

Constant Group Delay Filter

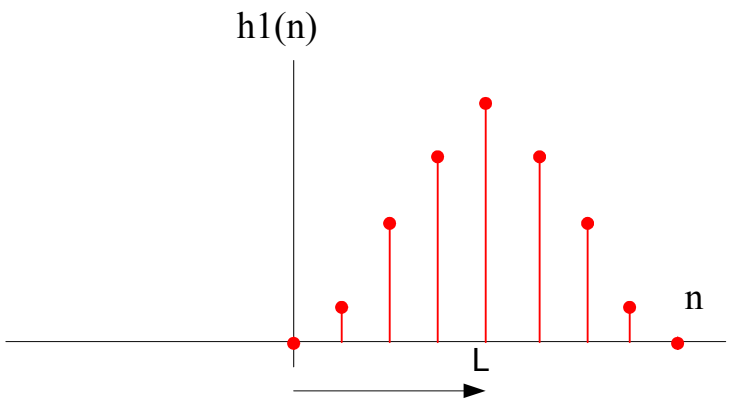
- Take even-symmetric FIR H(z) and shift its symmetry axis by L time samples

$$H_1(z) = z^{-L}\, H(z)$$

$$H_1(e^{j2\pi\Omega}) = e^{-j2\pi\Omega L}\, H(e^{j2\pi\Omega})$$

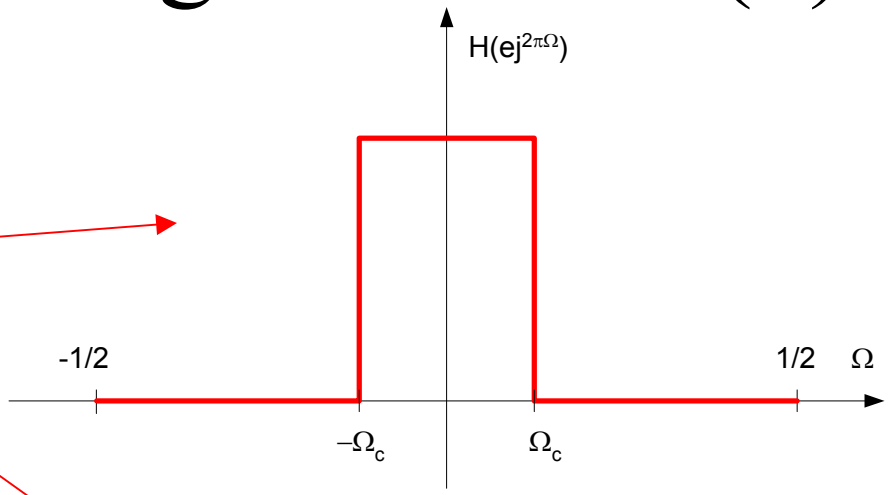


- Since $H(e^{j2\pi\Omega})$ is purely real, H_1 has a **linear phase characteristic** -> Constant Group Delay



Example 2: Design using Windows (1)

- Idea:
  - ✓ start from an **ideal frequency response** template (rectangular)
  - ✓ expand this function as an **infinite Fourier series** over the normalized frequency interval [-1/2,1/2]
  - ✓ identify the Fourier coefficients as the **FIR coefficients** h(n)
  - ✓ **truncate** this series to the desired number of FIR coefficients
  - ✓ multiply the impulse response by a **window** to limit the Gibbs phenomenon



$$H(e^{j2\pi\Omega}) = \sum_{n=-\infty}^{\infty} c_n . e^{-j2\pi n\Omega}$$

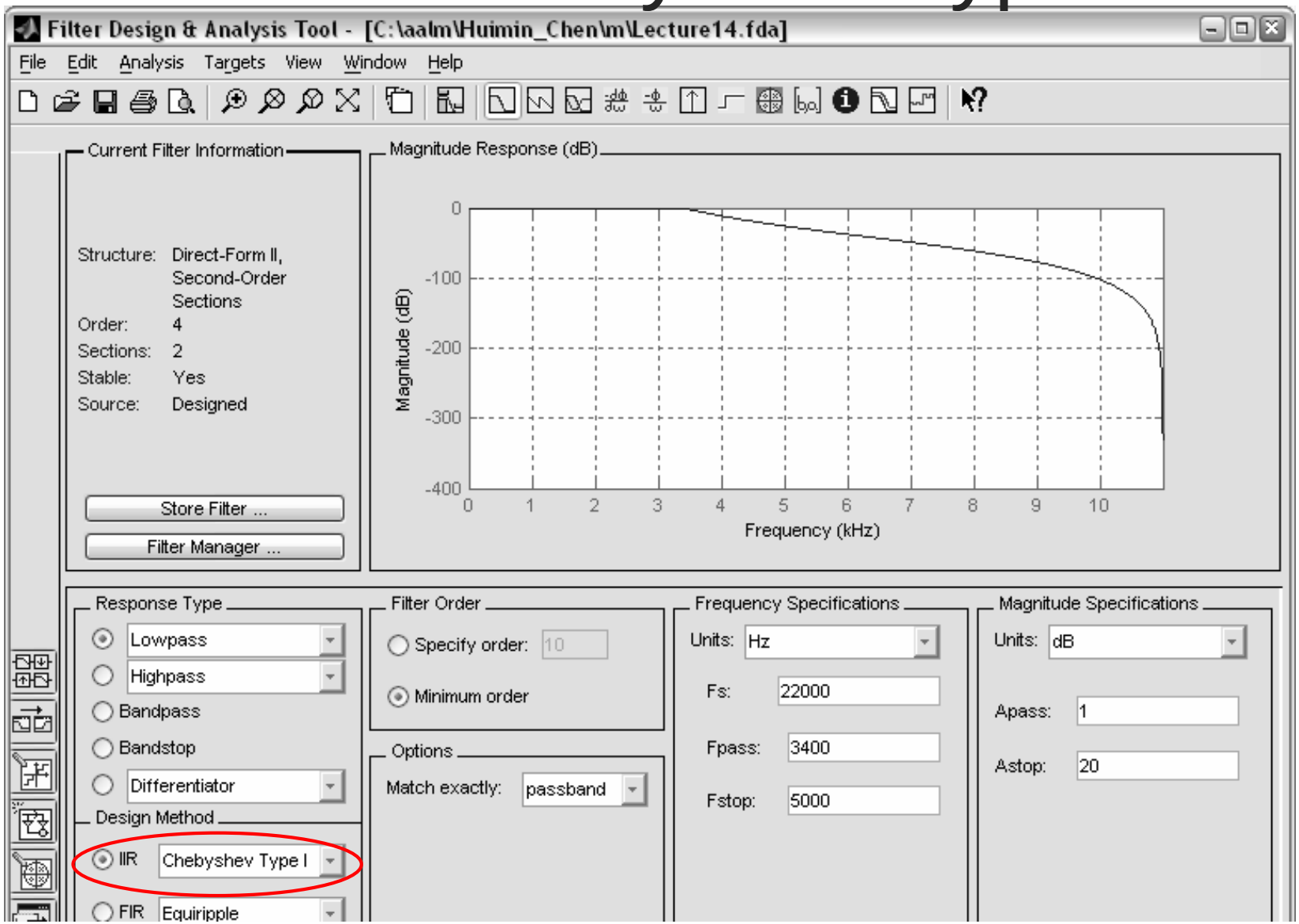
$$\text{with } c_n = \int_{-1/2}^{1/2} H(e^{j2\pi\Omega}).e^{j2\pi n\Omega}\, d\Omega$$

$$h(n) = c_n \quad \text{for } n \in [-\tfrac{N-1}{2}, \tfrac{N-1}{2}]$$

$$\text{and } h(n) = 0 \text{ elsewhere}$$

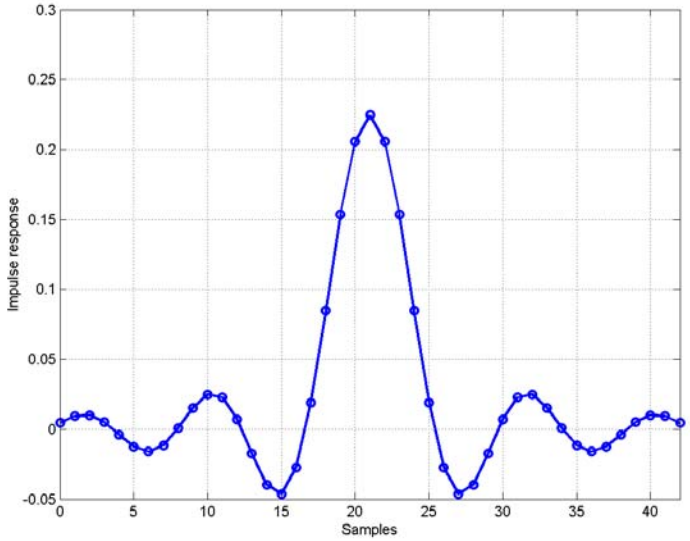
$$h'(n) = h(n) . w(n)$$

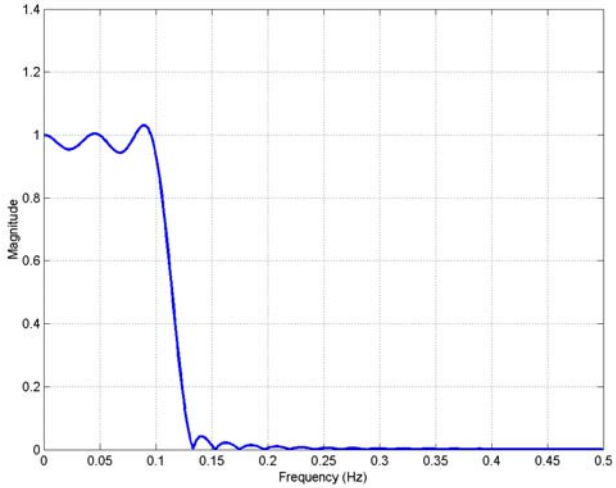
Step 1: Approximation
Select Chebyshev Type I



Example 2: Design using Windows (2)

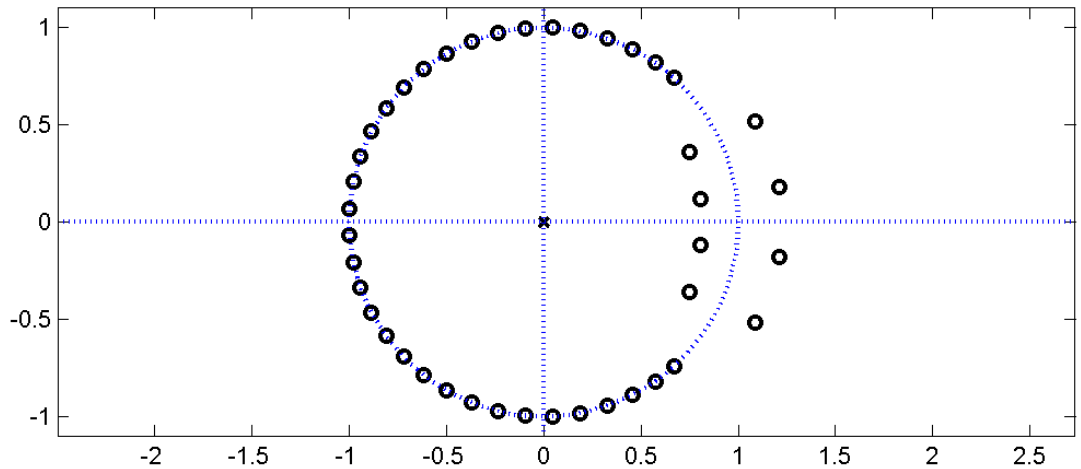
- Using **Kaiser window**
- Same spec as before:
  - ✓ Pass band End = 0.1
  - ✓ Pass band Ripple = 5%
  - ✓ Stop band Start = 0.13
  - ✓ Stop band Attenuation = 1/10
- Comparison with Equiripple design:
  - ✓ 43 coefficients vs. 31 (-)
  - ✓ Decreasing amplitude ripples in the Stop band (+)





Example 2: Design using Windows (3)

- Pole-Zero plot
- Location of zeros:
  - ✓ On the unit circle in the Stop band
  - ✓ Far from unit circle in Pass band



Needed number of coefficients

- For equiripple LP FIR filters:

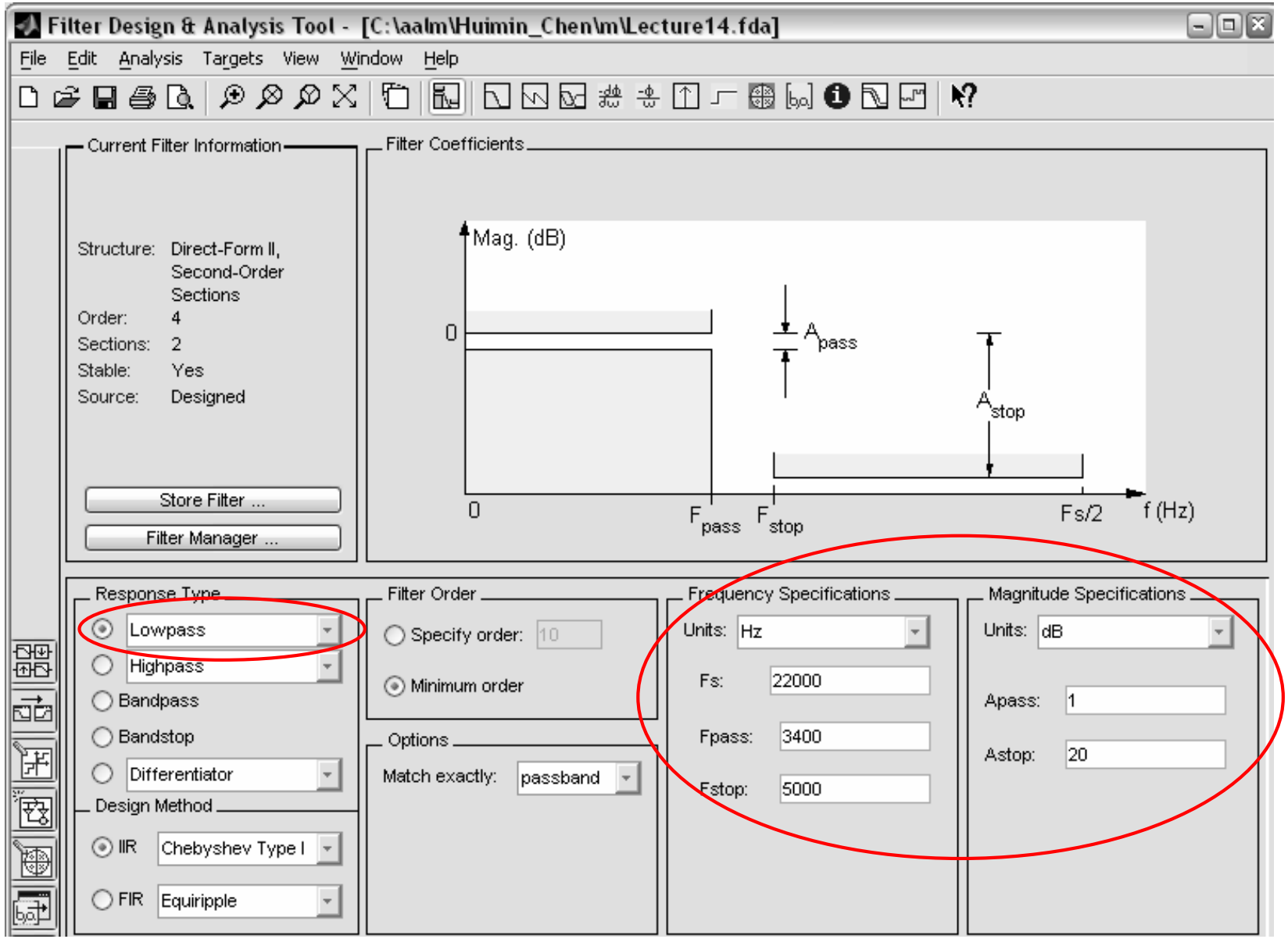
$$N_e = \frac{2}{3} \log\left[\frac{1}{10\, D_{pass}\, D_{stop}}\right] \frac{F_s}{F_{stop} - F_{pass}}$$

  - ✓Independent of BW (F_{pass})!
  - ✓Weak (logarithmic) dependence on the Pass band ripple level and the Stop band attenuation
  - ✓**Linear dependence on the transition band!**
- Our example: Ne = 29 (compared to 31)
- Problem: Very narrow filters -> **Decimating**

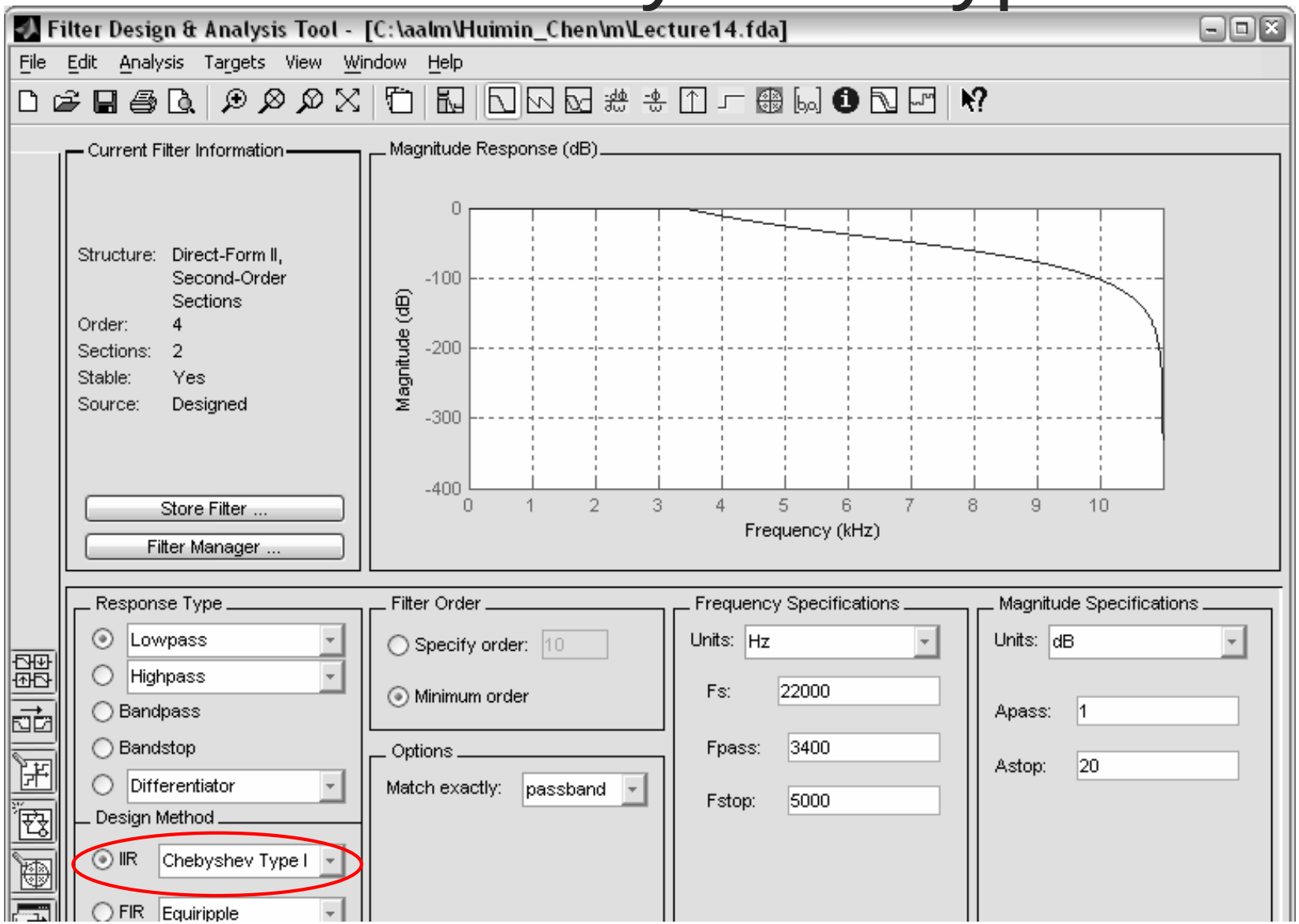
Overview

- Specification
- Step 1: Approximation - Select Chebyshev Type I
- Step 2: Realization - Cascade of biquads
- Step 3: Study of imperfections - Quantizations
- Redo design steps
  - Step 1: Approximation
  - Step 2: Realization
  - Step 3: Study of imperfections
- Step 4: Implementation
- MATLAB FDAtool documents

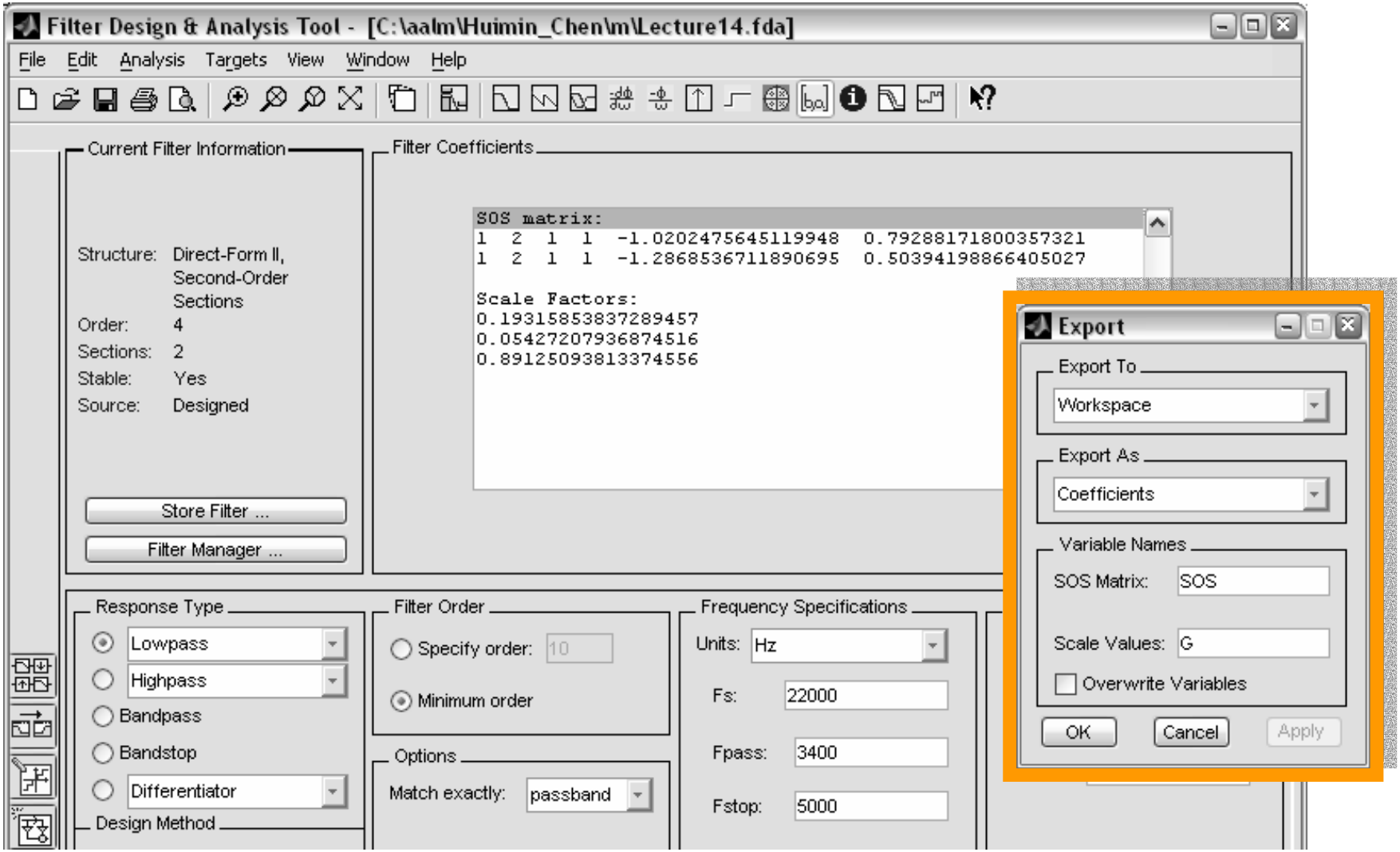
Start with specification



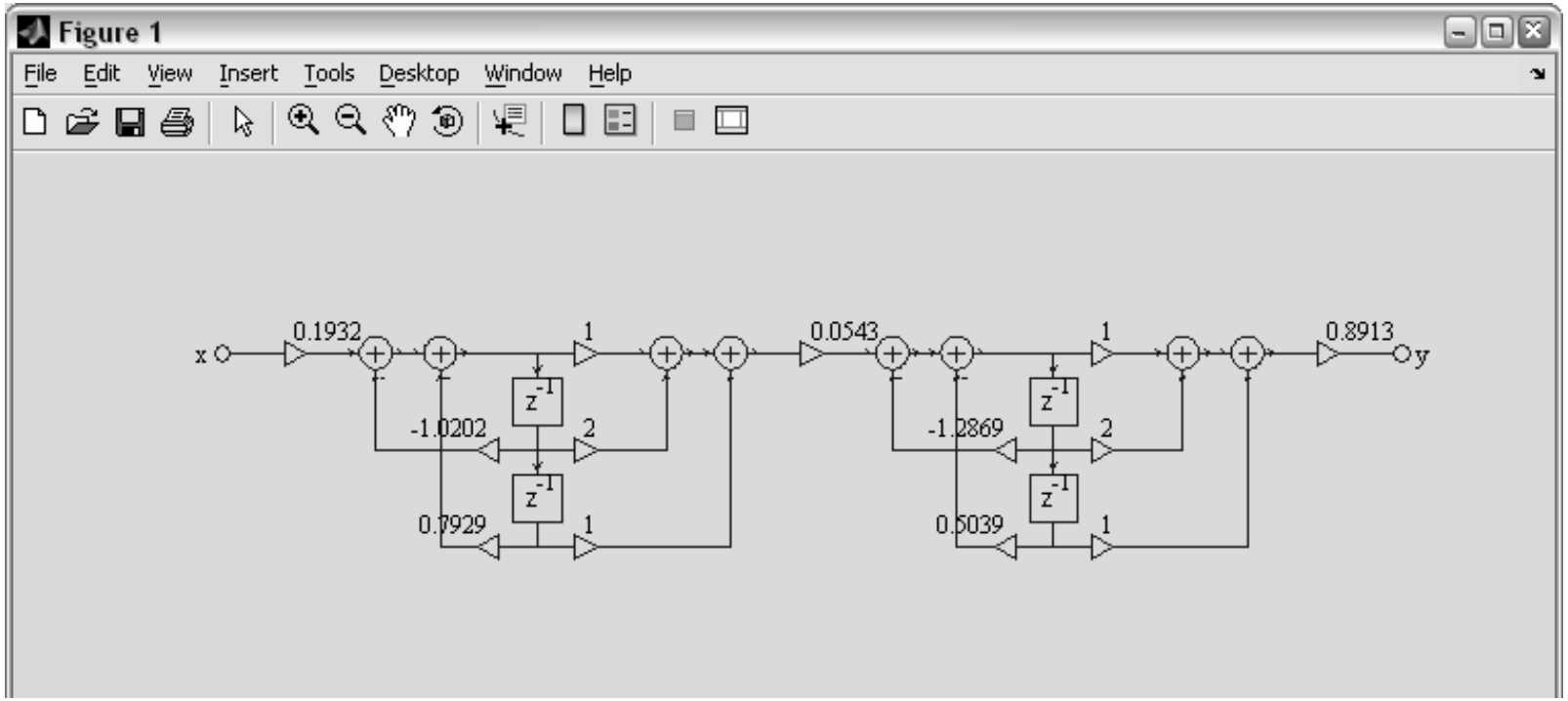
Step 1: Approximation
Select Chebyshev Type I



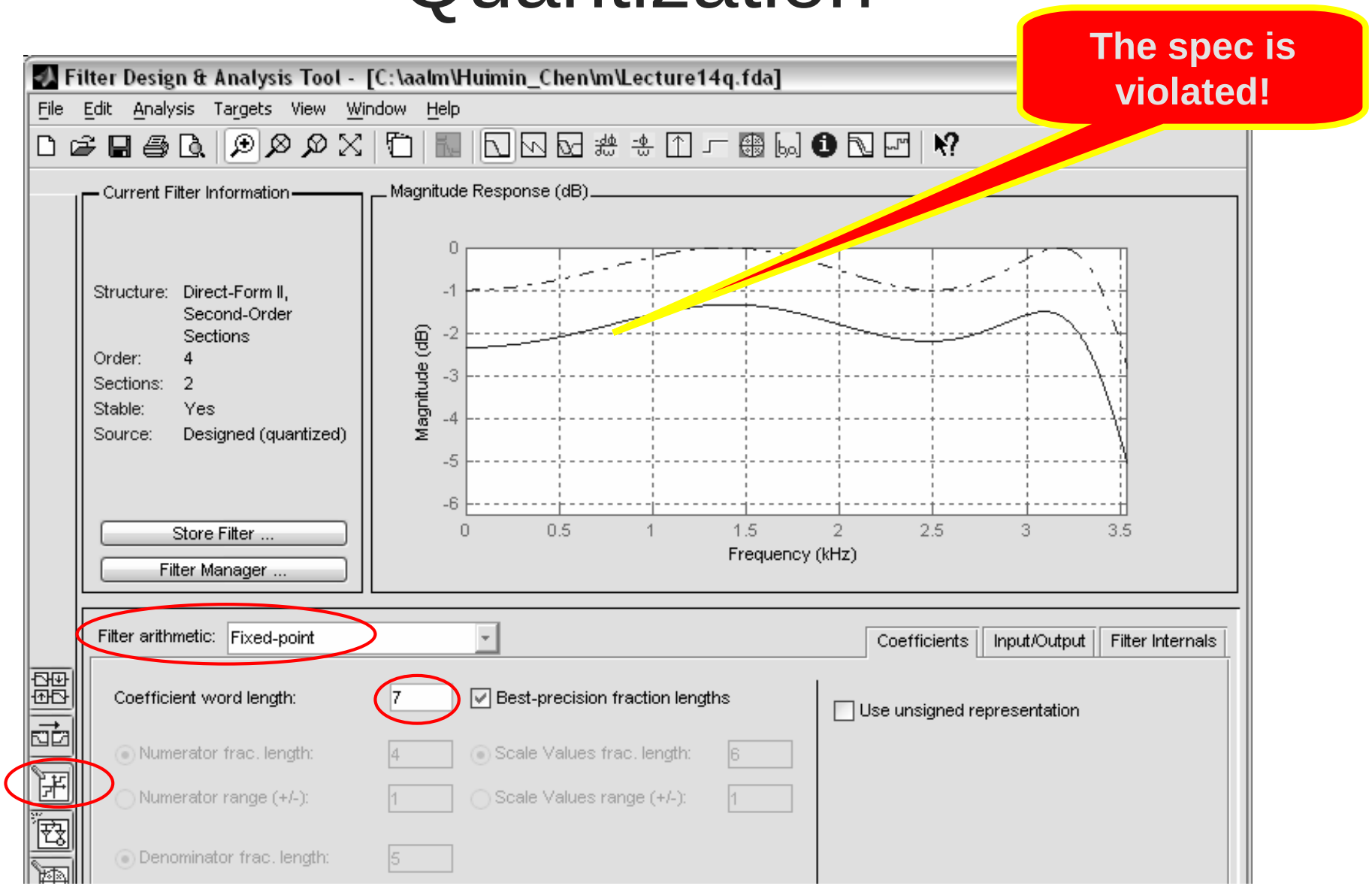
Export coefficients: **File, Export**



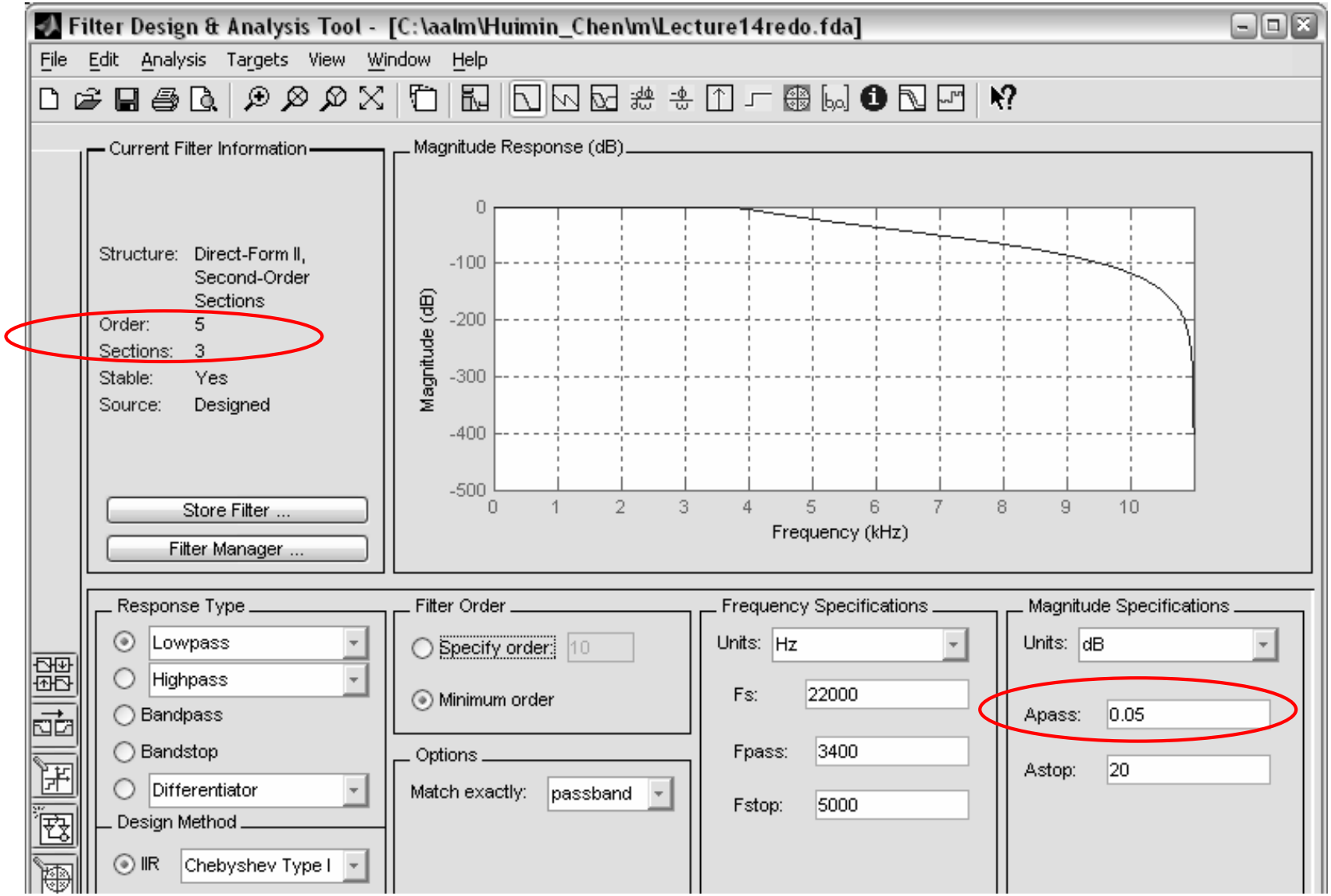
Step 2: Realization
Direct-form II biquads



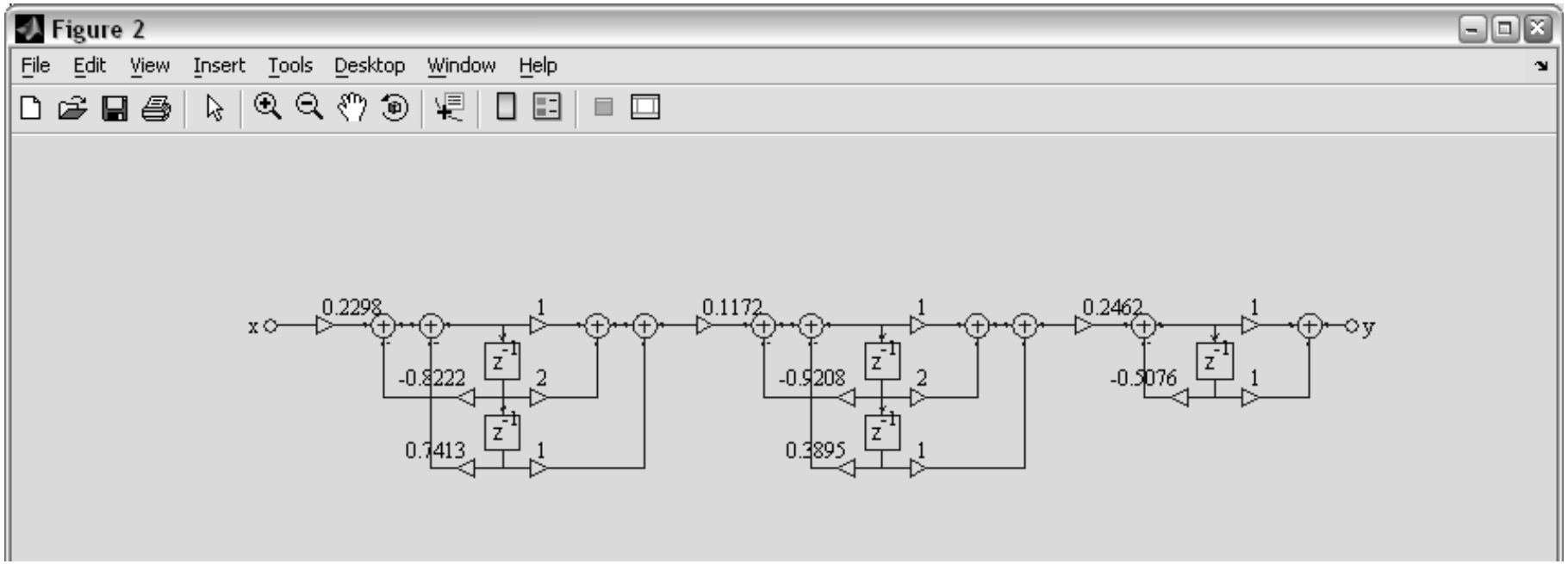
Step 3: Study of imperfections: Quantization



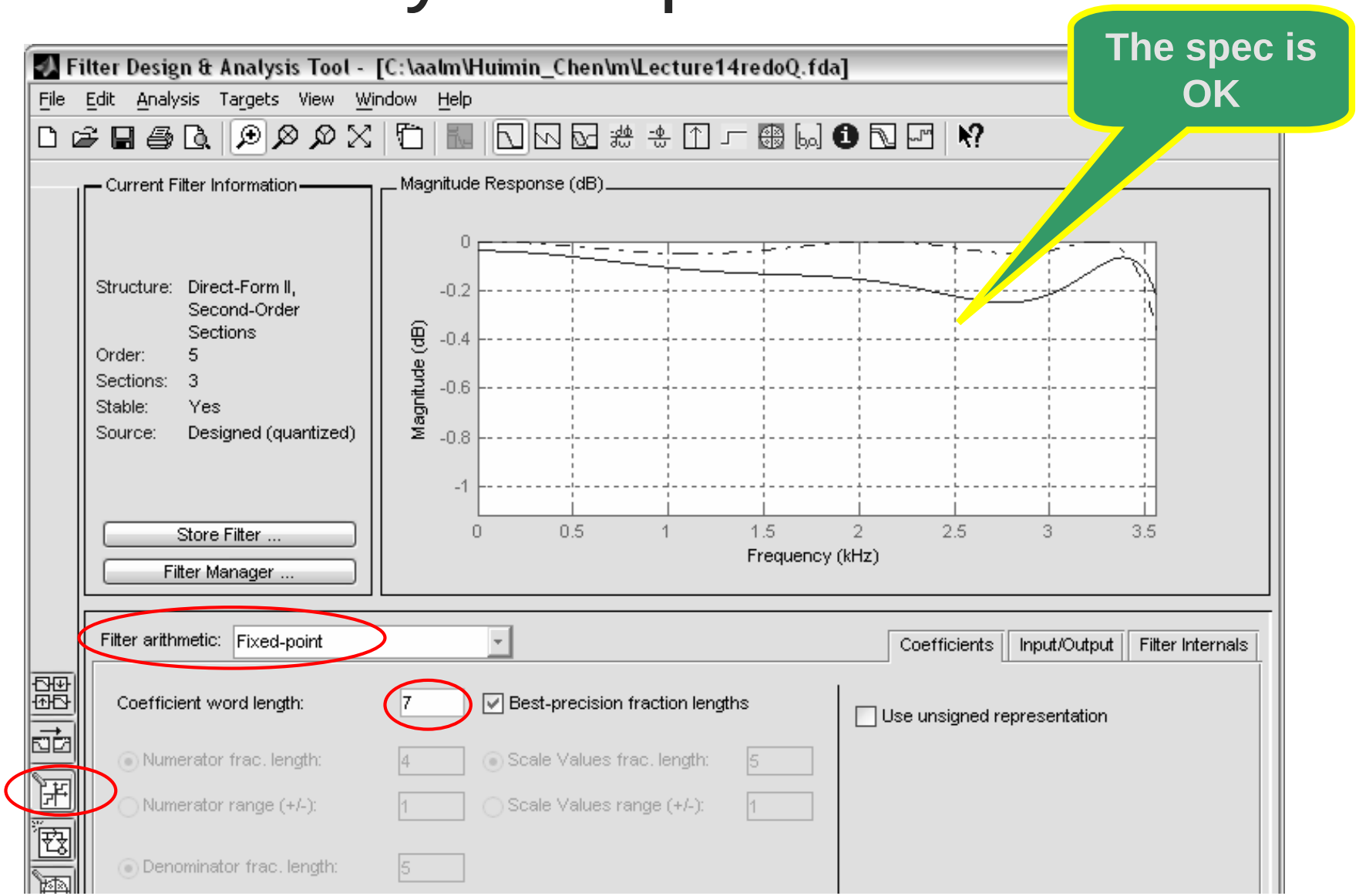
Redo step 1: Approximation
Increase order



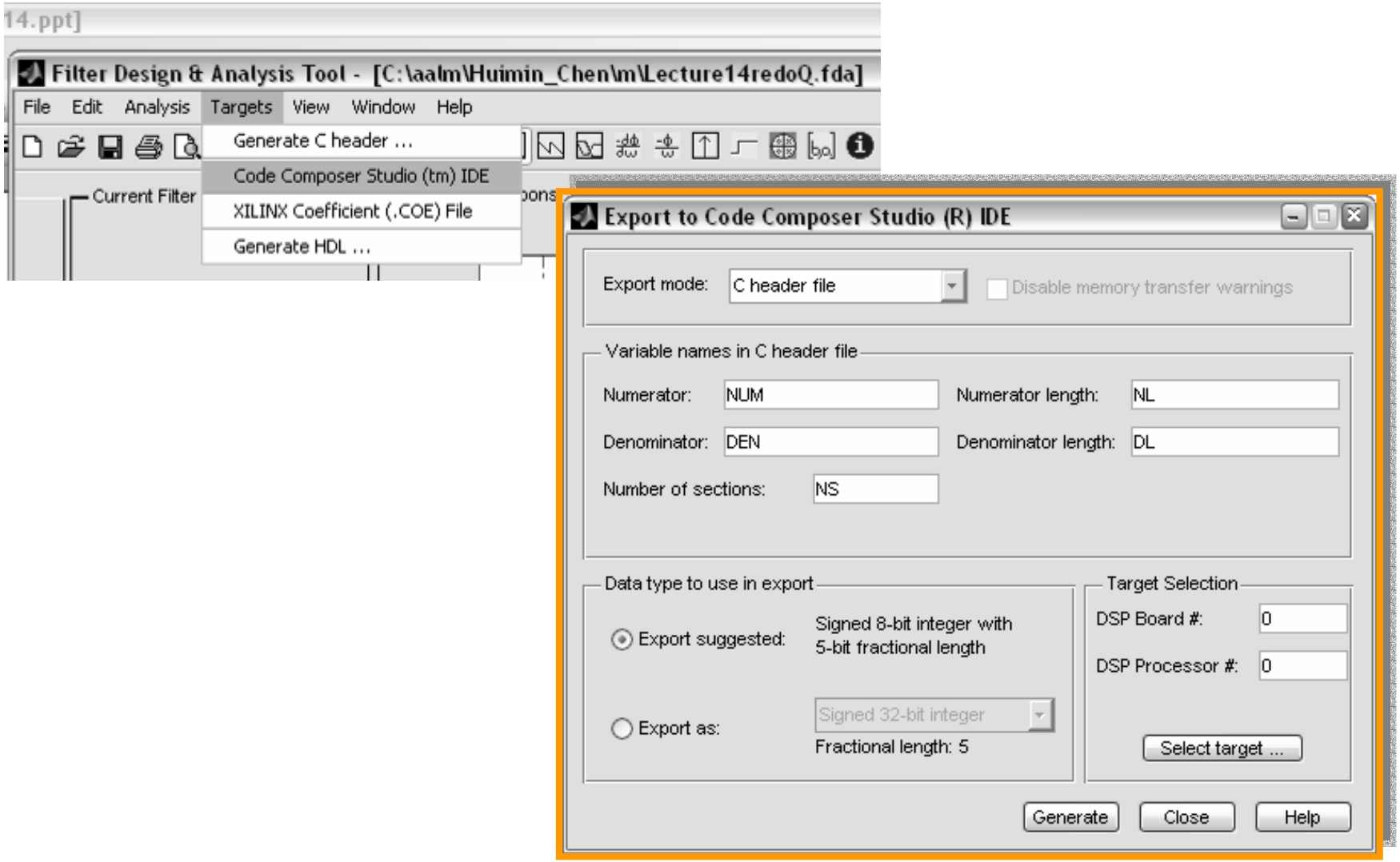
Redo step 2: Realization



Redo step 3:
Study of imperfections

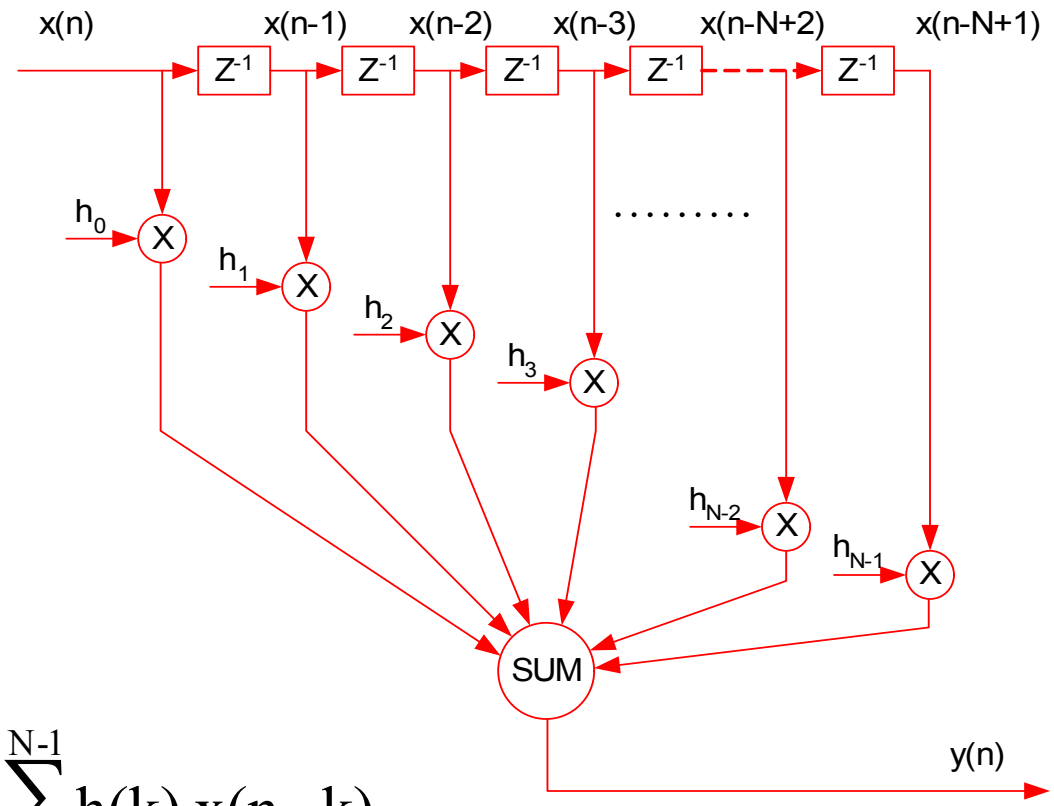


Step 4: Implementation



Basic Structure for FIR

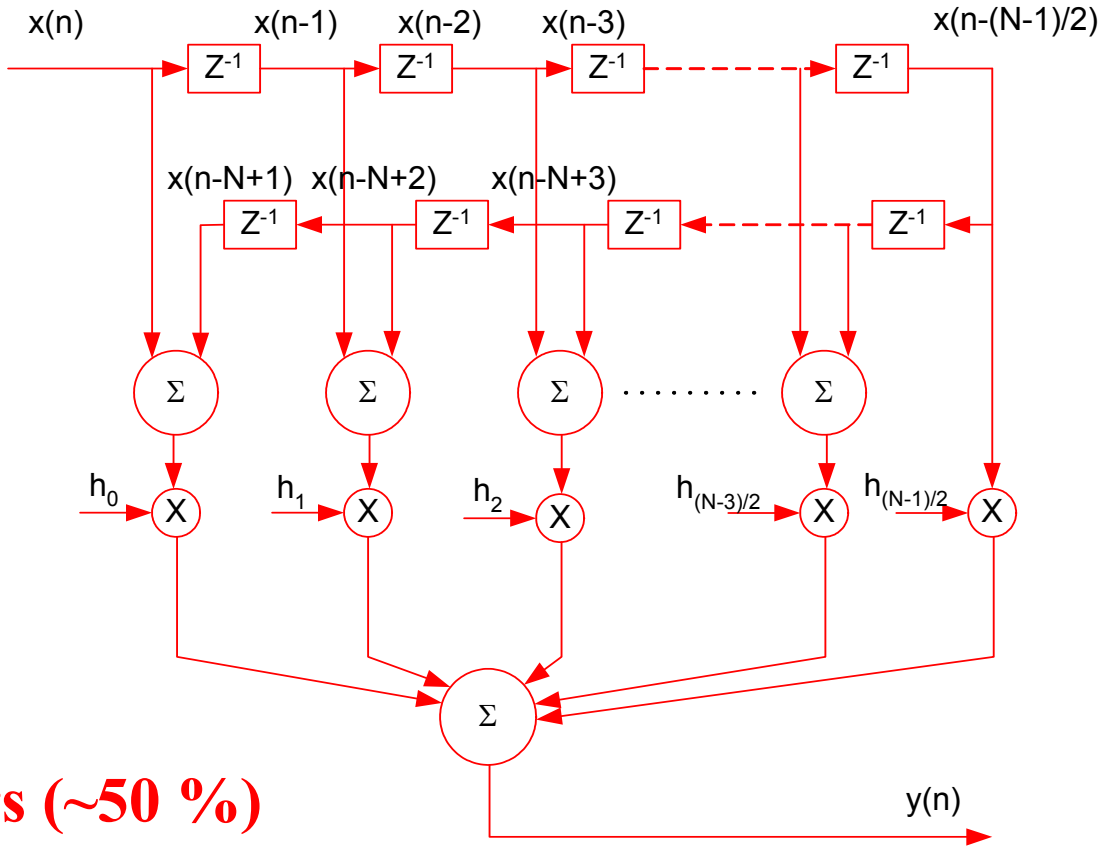
- Tapped-delay line (N-1) delays
- N multipliers
- 1 adder (N inputs)



$$y(n) = h(n) * x(n) = \sum_{k=0}^{N-1} h(k)\, x(n-k)$$

Structure for symmetric FIR

- Tapped-delay line (N-1) delays
- (N+1)/2 multipliers
- (N-1)/2 adders (2 inputs)
- 1 adder (N+1)/2 inputs



We save on multipliers (~50 %)

An hardware example: LF3320 (1)

LOGIC DEVICES INCORPORATED

LF3320
Horizontal Digital Image Filter

83 MSPS

introduced in 1998

FEATURES

- 83 MHz Data Rate
- 12-bit Data or Coefficients (Expandable to 24-bit)
- 32-Tap FIR Filter, Cascadable for More Filter Taps
- Over 49 K-bits of on-board Memory
- LF Interface™ Allows All 256 Coefficient Sets to be Updated Within Vertical Blanking
- Various Operating Modes: Dual Filter, Single Filter, Double Wide Data or Coefficient, Matrix Multiplication, and Accumulator Access.
- Selectable 16-bit Data Output with User-Defined Rounding and Limiting
- Supports Interleaved Data Streams
- Supports Decimation up to 16:1 for Increasing Number of Filter Taps
- 3.3 Volt Supply
- 144 Lead PQFP

DESCRIPTION

The **LF3320** filters digital images in the horizontal dimension at real-time video rates. The input and coefficient data are both 12 bits and in two's complement format. The output is also in two's complement format and may be rounded to 16 bits.

The LF3320 is designed to take advantage of symmetric coefficient sets. When symmetric coefficient sets are used, the device can be configured as a single 32-tap FIR filter or as two separate 16-tap FIR filters.

When asymmetric coefficient sets are used, the device can be configured as a single 16-tap FIR filter or as two separate 8-tap FIR filters. Multiple LF3320s can be cascaded to create larger filters.

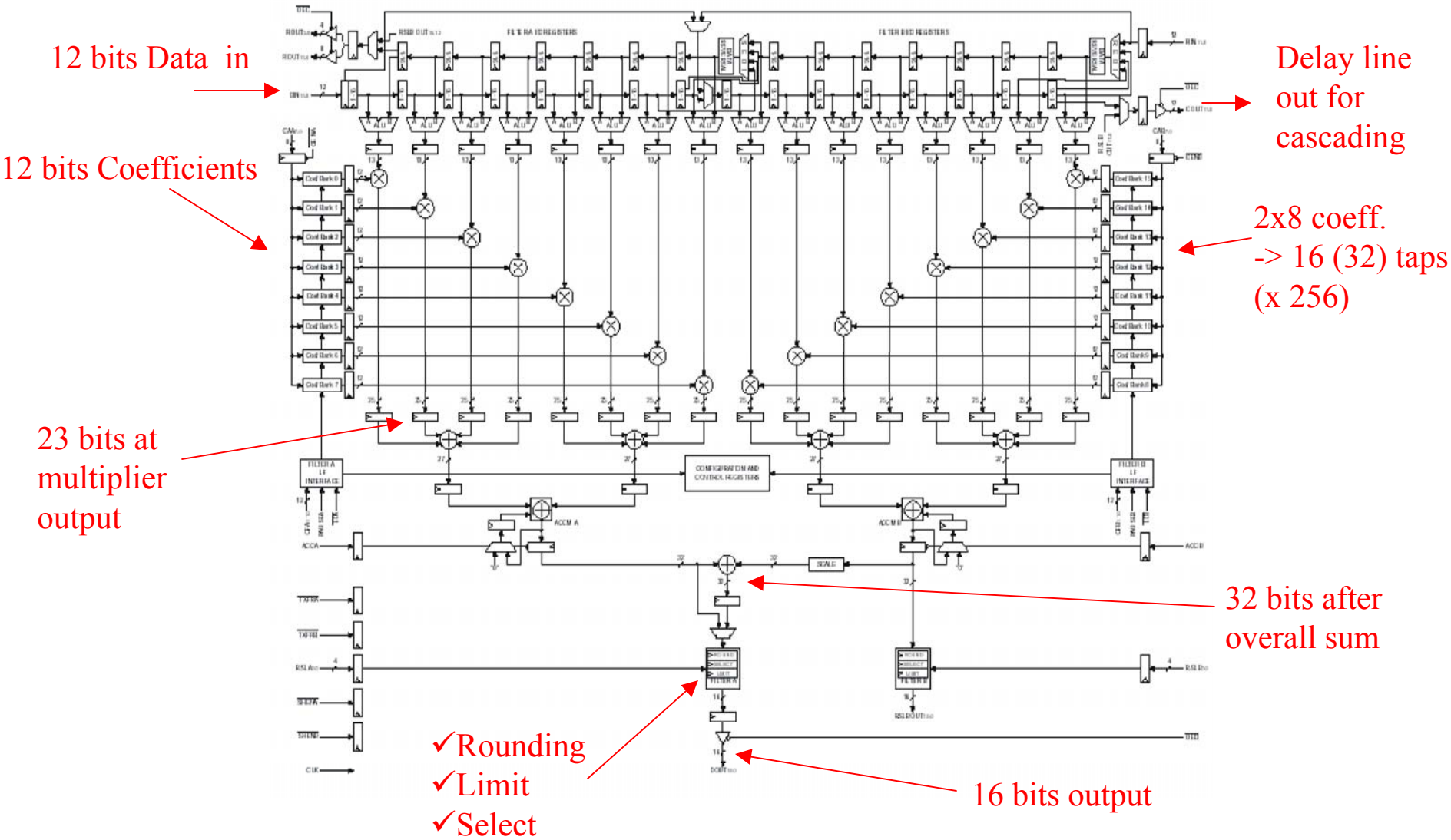
Interleave/Decimation Registers (I/D Registers) allow interleaved data to be fed directly into the device and filtered without separating the data into individual data streams.

The LF3320 can handle a maximum of sixteen data sets interleaved together. The I/D Registers and on-chip accumulators facilitate using decimation to increase the number of filter taps. Decimation of up to 16:1 is supported.

The LF3320 contains enough on-board memory to store 256 coefficient sets. Two separate LF Interfaces™ allow all 256 coefficient sets to be updated within vertical blanking.

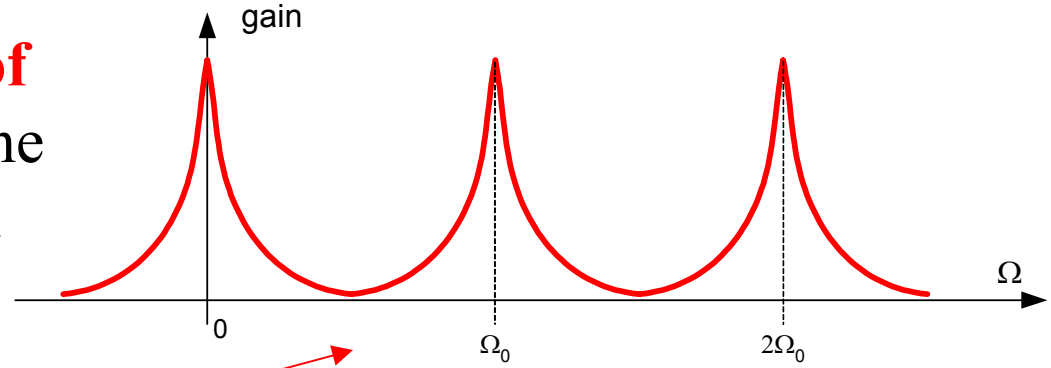
Used in the Feed-forward and Feed-back on the SPS 200 MHz cavities and in the SPS longitudinal damper.

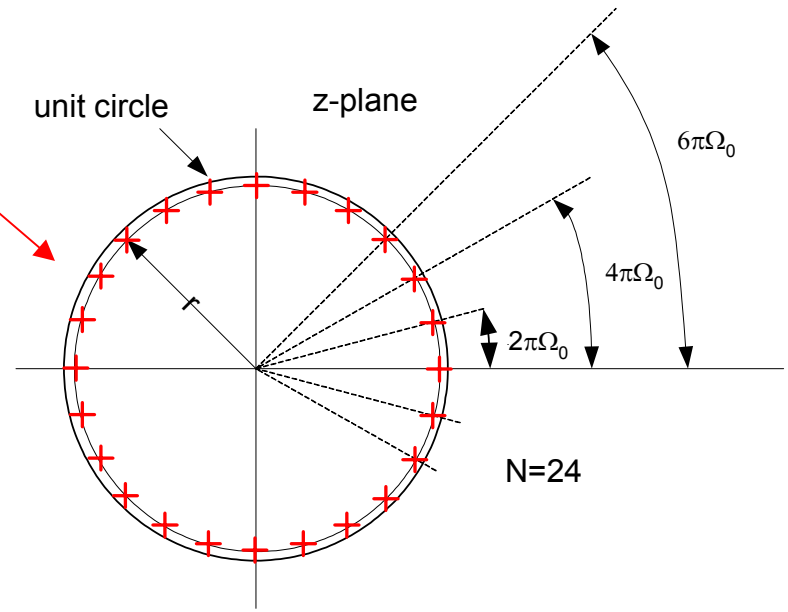
An hardware example: LF3320 (2)



IIR Design based on pole-zero plot

- Idea: Deduce the **location of the poles and zeros** from the desired frequency response.
- Example 3: Comb Filter
  - ✓ want **periodic resonances** at 0, Ω_0, 2 Ω_0, …, (N-1) Ω_0
  - ✓ realized by a series of **equispaced poles**, on a circle of radius r, and at angles 0, $2\pi\ \Omega_0$, $4\pi\ \Omega_0$, …, 2π (N–1) Ω_0





IIR Design based on Analog Prototype

- Idea: Transform an **analog prototype** (Butterworth, Chebyshev, Elliptic) into a **digital filter**
- The **transformation** from s-plane to z-plane **must**
  - ✓ Map the $[-j\pi F_s , +j\pi F_s]$ portion of the imaginary axis (s-plane) on the unit circle in the z-plane
  - ✓ Preserve stability
- The **analog features are kept**
  - ✓ Digital Butterworth are monotonic in both the Pass band and Stop band
  - ✓ Digital Chebyshev have ripple in the Pass band but are monotonic in the Stop band (or vice versa)
  - ✓ Digital Elliptic are equiripple in both Pass band and Stop band

Example 4: Elliptic LP IIR (1)

LPF example (as before):

- ✓ Pass band End $F_{pass} = 0.1$
- ✓ Pass band Ripple D_{pass}=0.05 (5%=0.45 dB)
- ✓ Stop band Start F_{stop}= 0.13
- ✓ Stop band Attenuation D_{stop} = 0.1 (=20dB)

- Specifications can be achieved with a minimum **fourth-order elliptic** filter
- Transfer function is the **ratio of two fourth-order** polynomials

$$H(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}}$$

Numerator:

0.108106225554593

-0.225674608761028

0.312898868850319

-0.225674608761028

0.108106225554593

Denominator:

1.000000000000000

-2.846667053723339

3.458441598722129

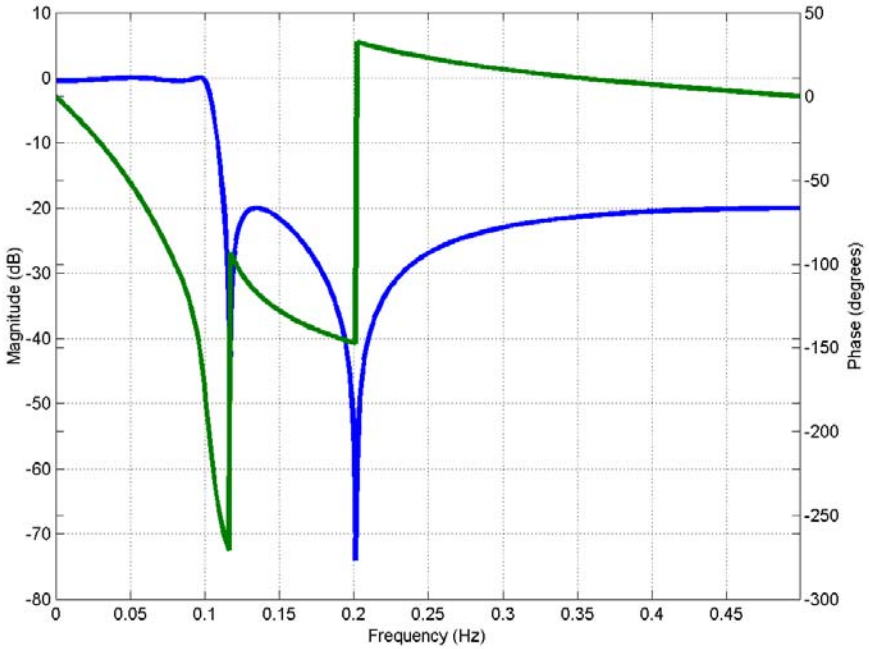
-2.013976284146102

0.484098739290319

Only 10 coefficients needed!!!

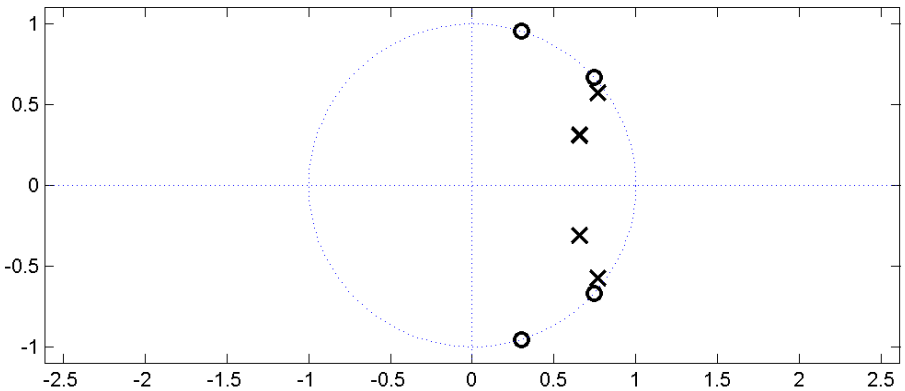
Example 4: Elliptic LP IIR (2)

- Achieved frequency response on **log scale** (blue trace)
- **Very non-linear** phase response (in green)
- Exact zeros in the stop band (Elliptic)



Example 4: Elliptic LP IIR (3)

- Pole-Zero plot in the z-plane
- **Poles** inside the unit circle (stability) at azimuth in the **Pass band**
- **Zeros** on the unit circle (elliptic) at azimuth in the **Transition band** and **Stop band**

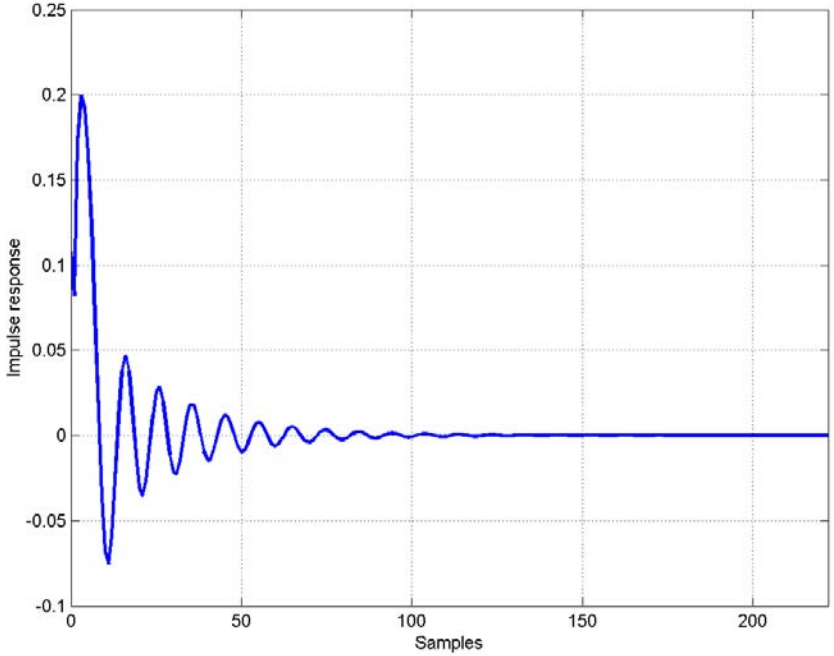


✓ *2* ***poles very close to the unit circle*** *(to get a steep transition band) -> caution !*

✓ ***Pole-zero cancellation*** *-> caution!*

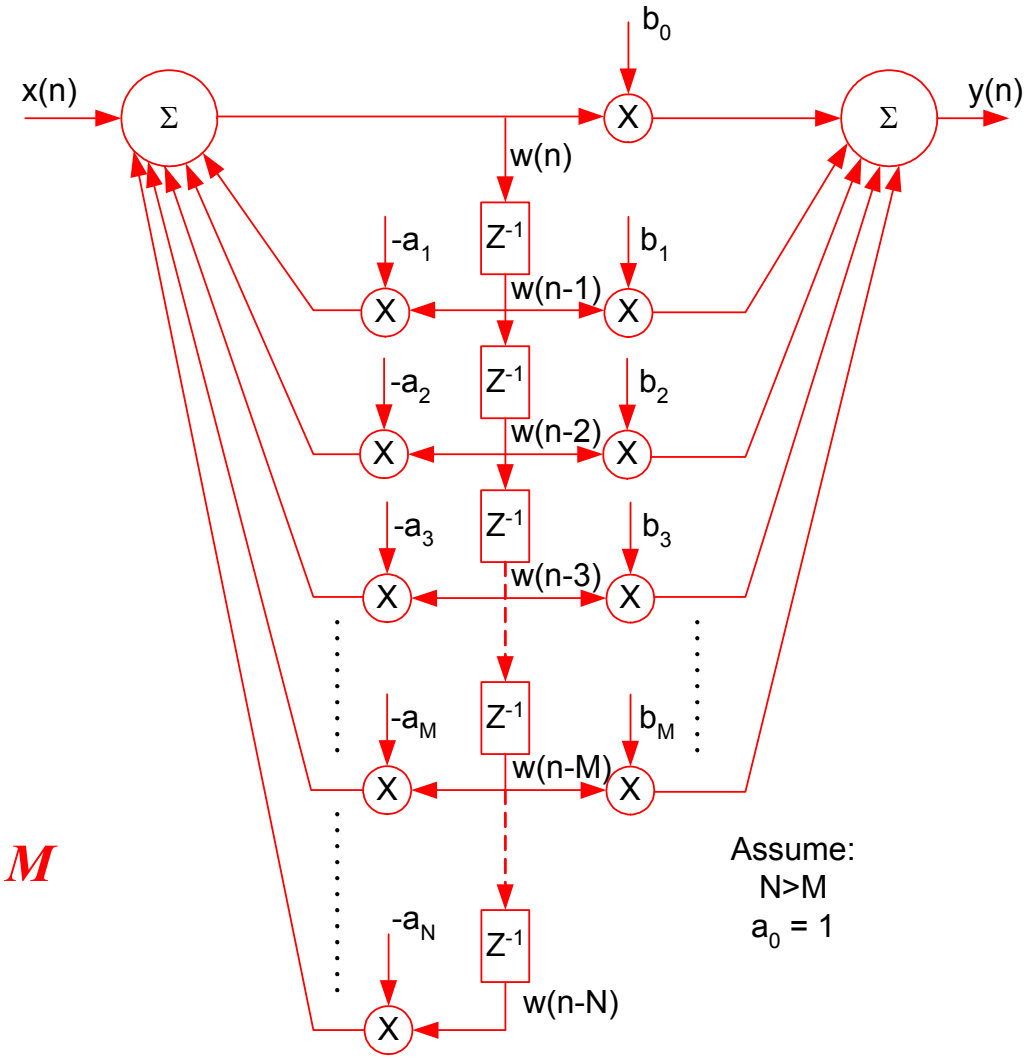
Example 4: Elliptic LP IIR (4)

- Impulse response lasts **forever** (IIR!)
- Comparison with FIR design:
  - ✓ Significant **reduction in the computational complexity:** 10 Multiply/Add compared to 31 (+)
  - ✓ **Very sensitive to quantization effects**. See next lecture (-)



Basic Structure for IIR

- Tapped-delay line (N or M) delays
- N + M + 1 multipliers
- 2 adders (N and M+1 inputs)
- **D/N structure** (or poles/zeros structure) also called Direct Form II



Very sensitive to the effects of coefficients quantization if N or M are large!!! (next lecture)

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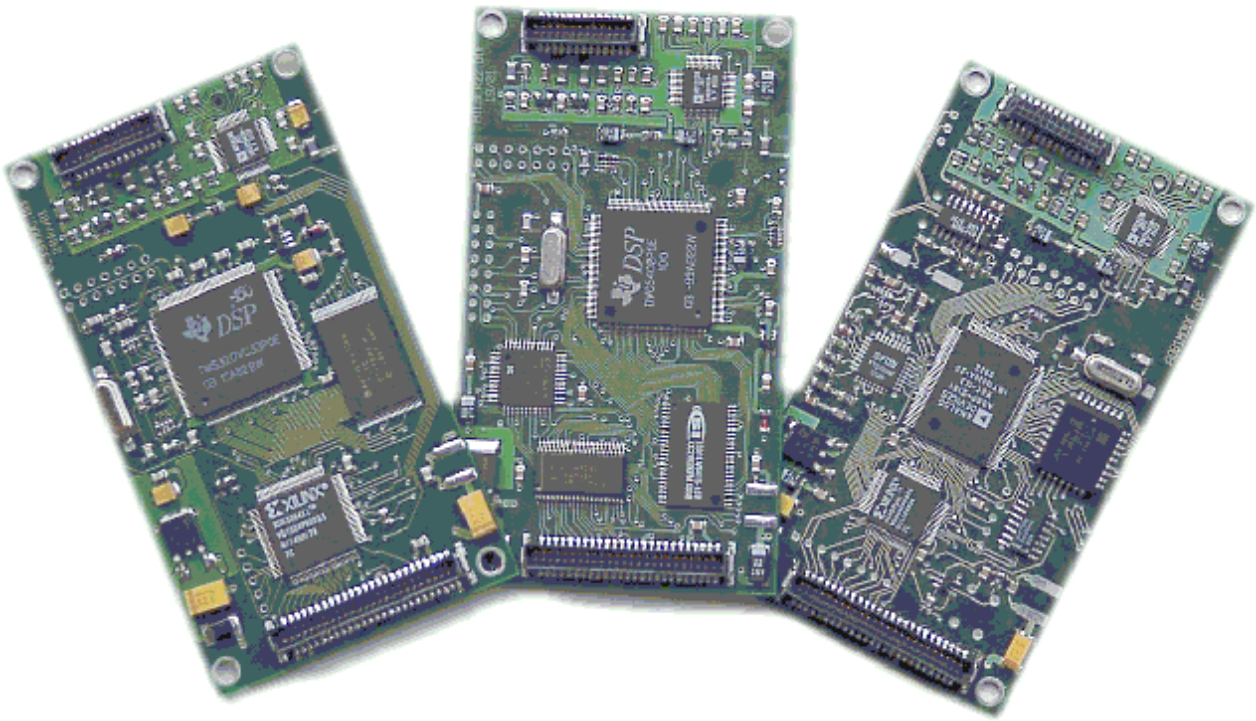

Hardware - Software - Algorithms
Consulting - Tools - OEM Products

TigerSHARC® Embedded Processor

ADSP-TS201S

KEY FEATURES

Up to 600 MHz, 1.67 ns instruction cycle rate

24M bits of internal—on-chip—DRAM memory

25 mm × 25 mm (576-ball) thermally enhanced ball grid array package

Dual-computation blocks—each containing an ALU, a multiplier, a shifter, a register file, and a communications logic unit (CLU)

Dual-integer ALUs, providing data addressing and pointer manipulation

Integrated I/O includes 14-channel DMA controller, external port, four link ports, SDRAM controller, programmable flag pins, two timers, and timer expired pin for system integration

1149.1 IEEE compliant JTAG test access port for on-chip emulation

On-chip arbitration for glueless multiprocessing

KEY BENEFITS

Provides high-performance Static Superscalar DSP operations, optimized for telecommunications infrastructure and other large, demanding multiprocessor DSP applications

Performs exceptionally well on DSP algorithm and I/O benchmarks (see benchmarks in Table 1)

Supports low overhead DMA transfers between internal memory, external memory, memory-mapped peripherals, link ports, host processors, and other (multiprocessor) DSPs

Eases DSP programming through extremely flexible instruction set and high-level-language friendly DSP architecture

Enables scalable multiprocessing systems with low communications overhead

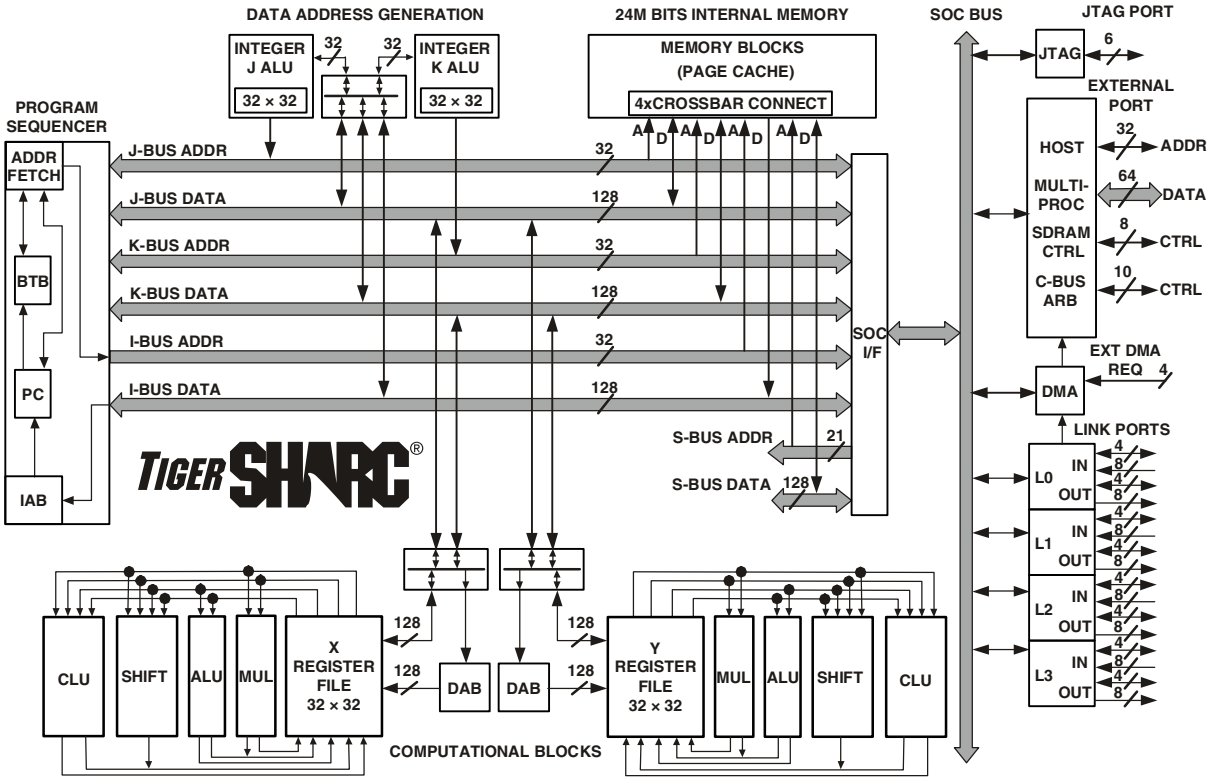

Figure 1. Functional Block Diagram

Rev. 0

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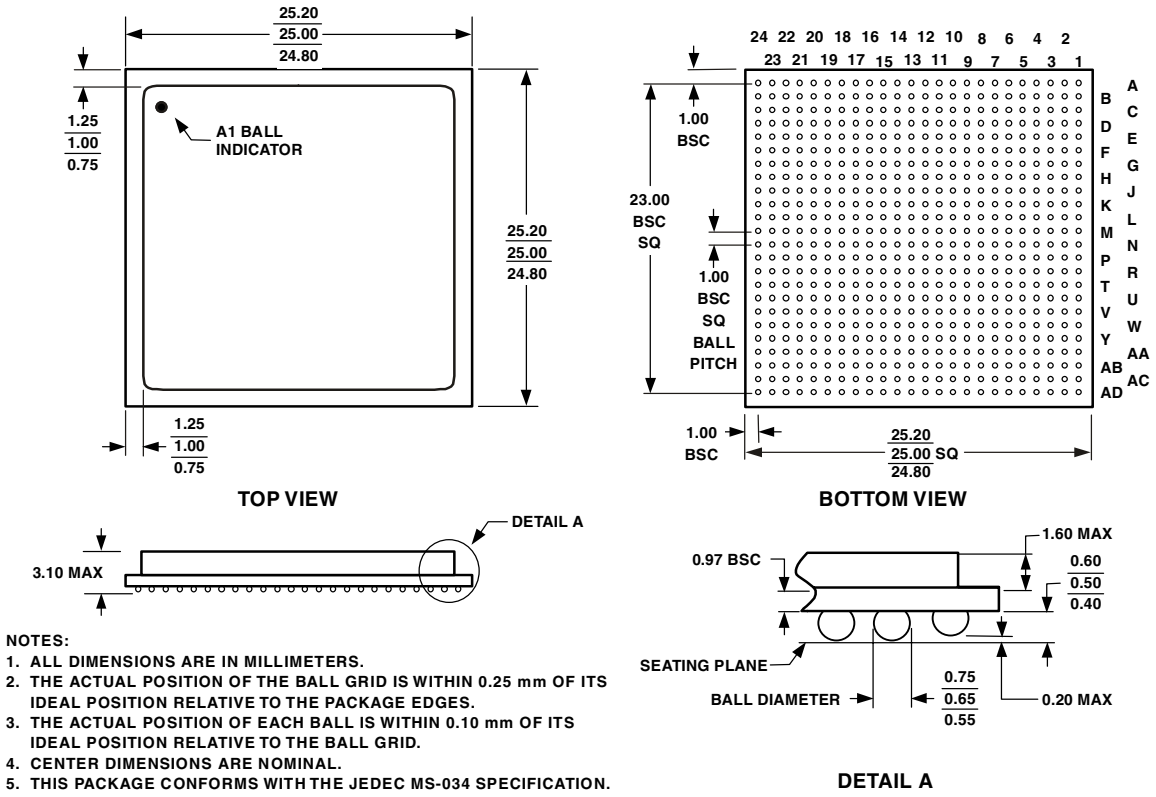
Tel: 781/329-4700 **www.analog.com**
Fax: 781/326-8703

OUTLINE DIMENSIONS

The ADSP-TS201S processor is available in a 25 mm × 25 mm, 576-ball metric thermally enhanced ball grid array (BGA_ED) package with 24 rows of balls (BP-576).



NOTES:
1. ALL DIMENSIONS ARE IN MILLIMETERS.
2. THE ACTUAL POSITION OF THE BALL GRID IS WITHIN 0.25 mm OF ITS IDEAL POSITION RELATIVE TO THE PACKAGE EDGES.
3. THE ACTUAL POSITION OF EACH BALL IS WITHIN 0.10 mm OF ITS IDEAL POSITION RELATIVE TO THE BALL GRID.
4. CENTER DIMENSIONS ARE NOMINAL.
5. THIS PACKAGE CONFORMS WITH THE JEDEC MS-034 SPECIFICATION.

Figure 46. 576-Ball BGA_ED (BP-576)

ORDERING GUIDE

Part Number[1, 2, 3, 4, 5]	Case Temperature Range	Instruction Rate[6]	On-Chip DRAM	Operating Voltage	Package
ADSP-TS201SABP-060	–40°C to +85°C	600 MHz	24M bit	1.20 V_{DD}, 2.5 V_{DD_IO}, 1.6 V_{DD_DRAM}	(BP-576)[7]
ADSP-TS201SABP-050	–40°C to +85°C	500 MHz	24M bit	1.05 V_{DD}, 2.5 V_{DD_IO}, 1.5 V_{DD_DRAM}	(BP-576)
ADSP-TS201SWBP-050	–40°C to +105°C	500 MHz	24M bit	1.05 V_{DD}, 2.5 V_{DD_IO}, 1.5 V_{DD_DRAM}	(BP-576)

[1] S indicates 1.xx/2.5 V supplies.
[2] A indicates –40°C to +85°C temperature.
[3] W indicates –40°C to +105°C temperature.
[4] BP indicates thermally enhanced ball grid array (BGA_ED) package.
[5] -060 indicates 600 MHz operation, and -050 indicates 500 MHz operation.
[6] The instruction rate is the same as the internal processor core clock (CCLK) rate.
[7] The BP-576 package measures 25 mm × 25 mm.

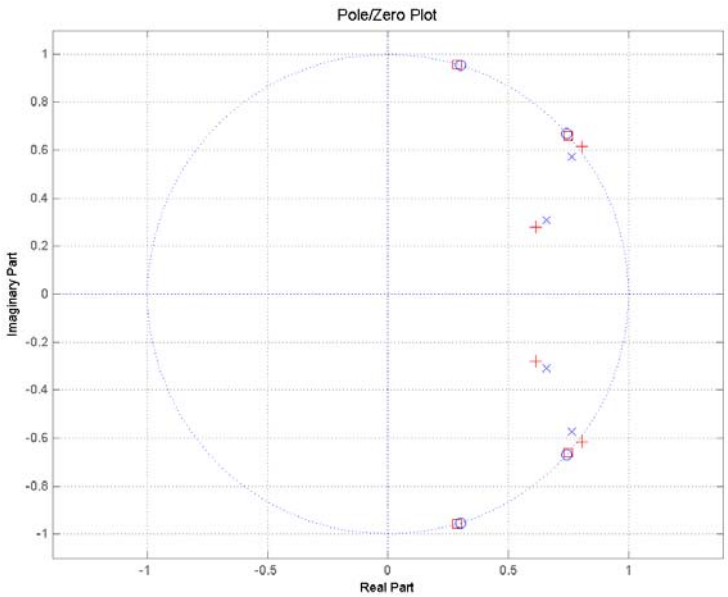
Quantization of coefficients

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_r z^{-r}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The **coefficients** a_k and b_r must be **quantized** into our two's complement fractional
- This creates a **distortion** in the achieved frequency response
- We measure the **sensitivity** to coefficient quantization by **comparing the poles/zeros** of the infinite precision and finite precision realizations

Elliptic IIR. Direct Form (1)

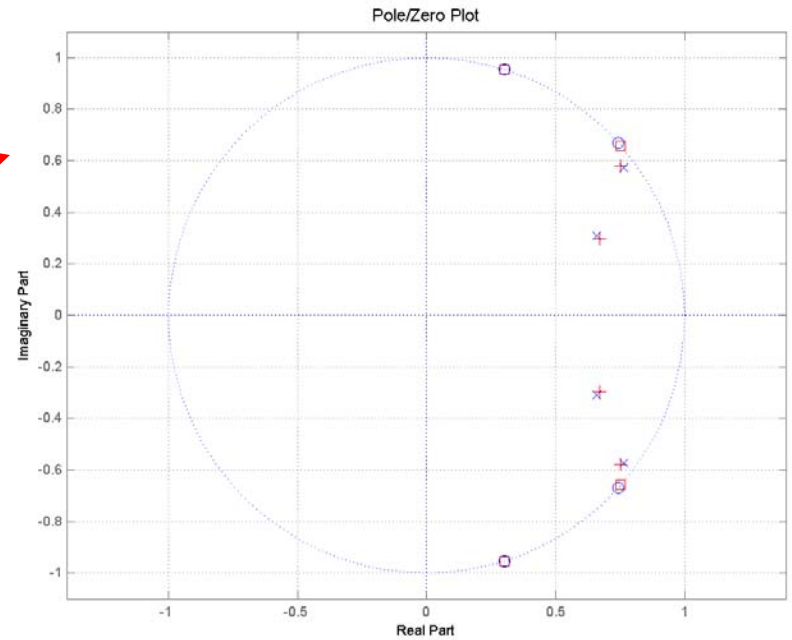
- Consider the **fourth order Elliptic** IIR LPF (Example 4, slide 11 in lecture 5, Part II)
- **Direct Form II** implementation (slide 15 in lecture 5, Part II)
- Quantize its coefficients with **8 bits** …
- … and the filter becomes ***unstable***. Two poles are moved outside the unit circle

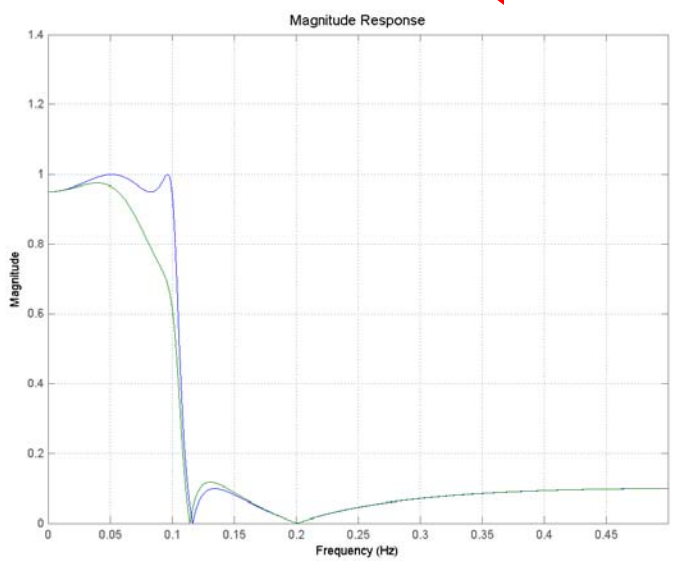


Pole/zero plot of the **infinite precision (blue) and finite precision (red)** Elliptic IIR. Direct Form II, 8 bits coefficients.

Elliptic IIR. Direct Form (2)

- … so we increase the number of bits: **9 bits** coefficients …
- … and it is **stable** but the frequency response shows **severe distortion**

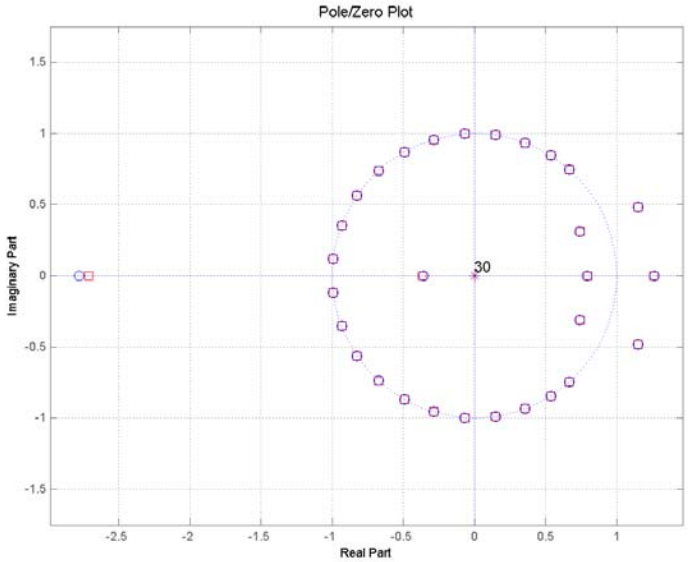


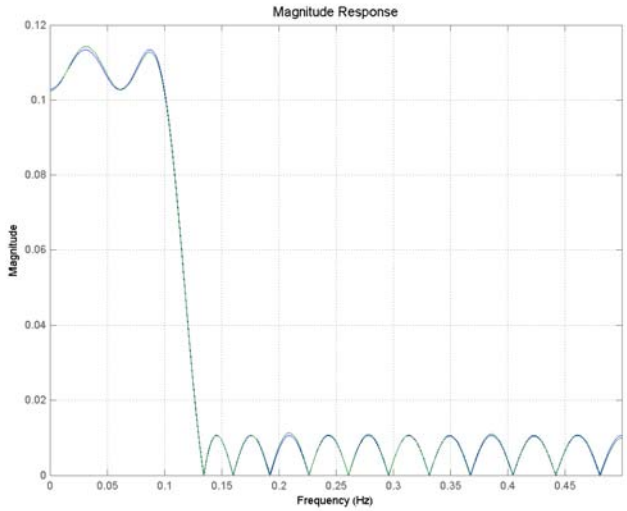


Pole/zero plot of the **infinite precision (blue)** and **finite precision (red)** Elliptic IIR.
Comparison of the **achieved frequency response (green) with the reference (blue).** Direct Form II, 9 bits coefficients.

Equiripple FIR (1)

- Coefficient quantization displaces the zeros as well and the **FIR response** will also suffer distortion
- Consider the **31 coefficients Equiripple FIR** (slide 17, lecture 5, Part I)
- First use **8 bits** coefficients. The result is **quite good**

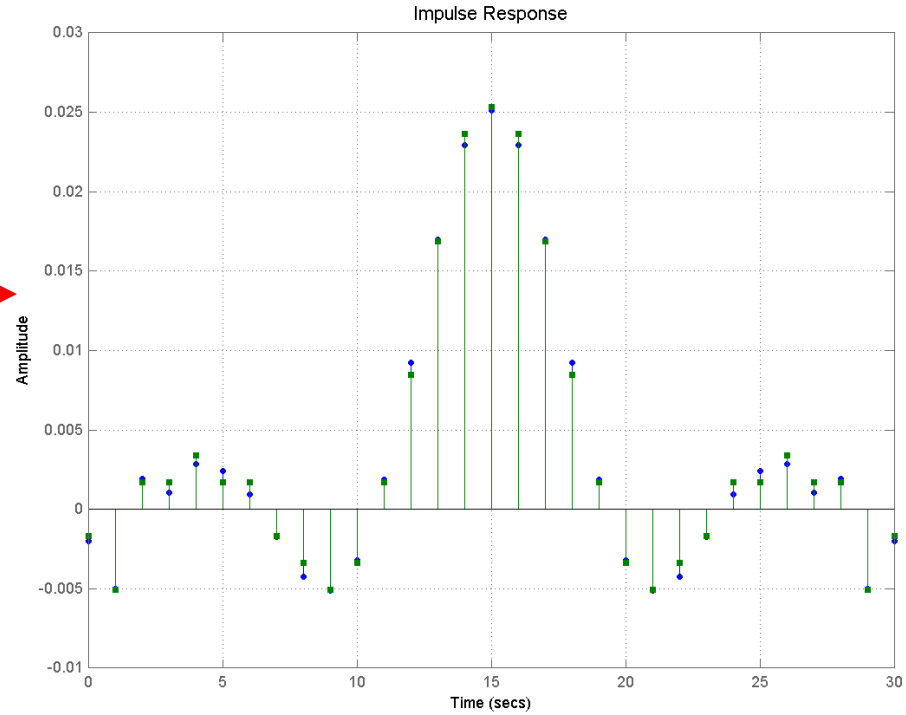


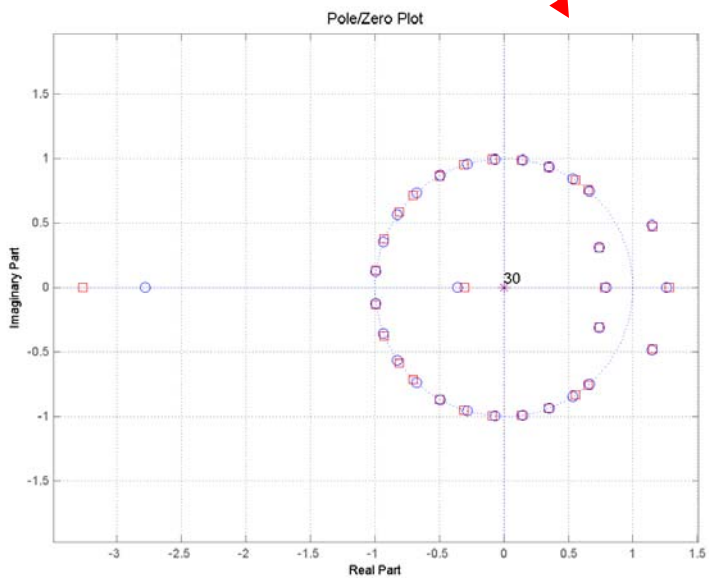


Pole/zero plot of the **infinite precision (blue)** and **finite precision (red)** Equiripple FIR. Comparison of the **achieved frequency response (green)** with the reference (blue). 8 bits coefficients.

Equiripple FIR (2)

- So we decrease the resolution further: **5 bits coefficients**…
- … the impulse response is much distorted …
- … the zeros are not displaced much …

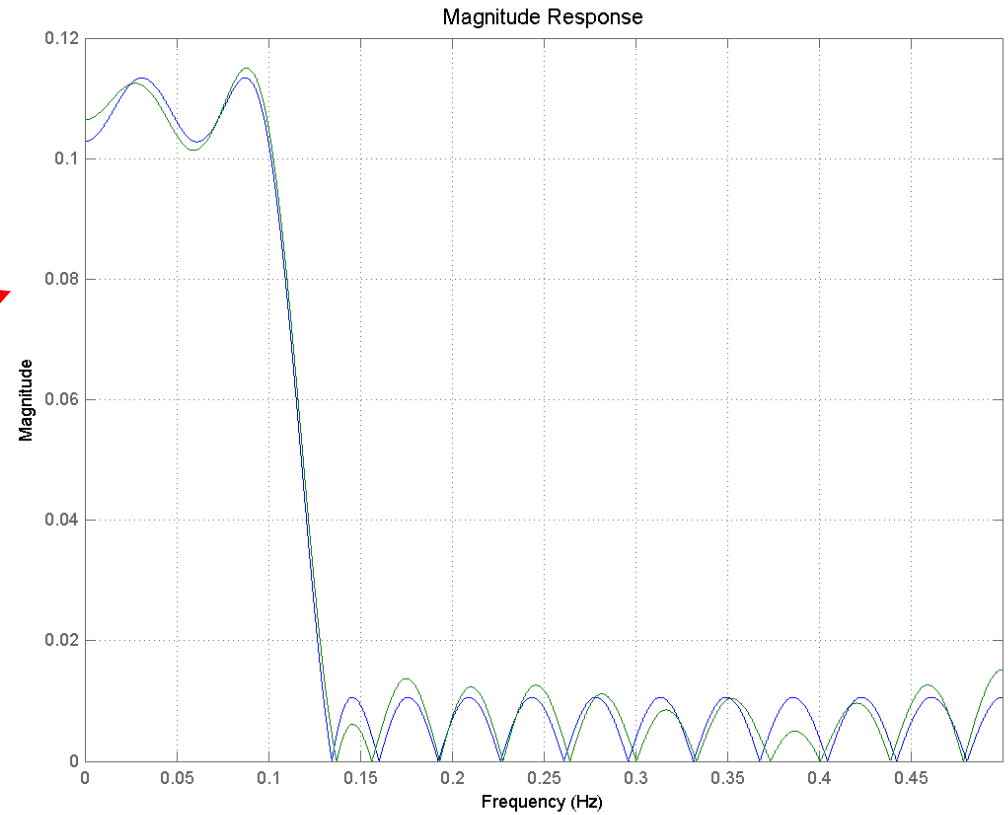




Impulse response of **infinite precision (blue dots)** and finite precision (green squares). Pole/zero plot of the **infinite precision (blue)** and **finite precision (red).** Equiripple FIR. 5 bits coefficients

Equiripple FIR (3)

- … and the frequency response is **still acceptable. It is actually almost within specs …**



Comparison of the **achieved frequency response (green)** with the reference (blue). Equiripple FIR. 5 bits coefficients.

Lesson: FIR are much less sensitive to coefficient quantization