BMT 403 :
Einführung in die Biosignalverarbeitung
Ergänzende Folien 3

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z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  leads to the *z*-transform
- *z*-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of *z*-transform techniques permits simple algebraic manipulations

z-Transform

- Consequently, *z*-transform has become an important tool in the analysis and design of digital filters
- For a given sequence *g*[*n*], its *z*-transform *G*(*z*) is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

  where $z = \text{Re}(z) + j\,\text{Im}(z) \in \mathbb{C}$ is a complex variable

z-Transform

- If we let $z = r e^{j\omega}$, then the z-transform reduces to

$$G(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

- The above can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$
- For r = 1 (i.e., |z| = 1), z-transform reduces to its DTFT, provided the latter exists

z-Transform

- The contour |z| = 1 is a circle in the z-plane of unity radius and is called the ***unit circle***
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set R of values of z for which its z-transform converges is called the ***region of convergence (ROC)***

z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

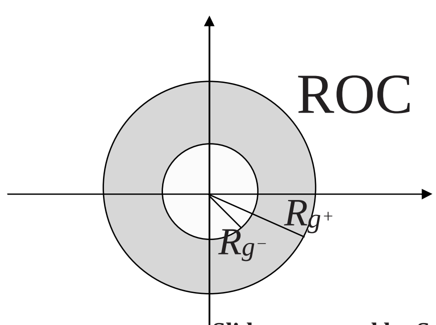
$$\sum_{n=-\infty}^{\infty} \left| g[n]r^{-n} \right| < \infty$$

z-Transform

- In general, the ROC of a *z*-transform of a sequence *g*[*n*] is an annular region of the *z*-plane:

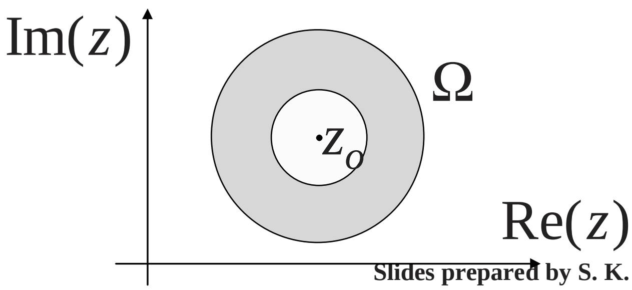
$$R_{g^-} < |z| < R_{g^+}$$

where $0 \le R_{g^-} < R_{g^+} \le \infty$



Cauchy-Laurent Series

- The z-transform is a form of the ***Cauchy-Laurent series*** and is an analytic function at every point in the ROC
- Let $f(z)$ denote an analytic (or holomorphic) function over an annular region Ω centered at z_o



Cauchy-Laurent Series

- Then $f(z)$ can be expressed as the bilateral series

$$f(z) = \sum_{n=-\infty}^{\infty} \alpha_n (z - z_o)^n$$

where

$$\alpha_n = \frac{1}{2\pi j} \oint_{\gamma} f(z)(z - z_o)^{-(n+1)} dz$$

γ being a closed and counterclockwise integration contour contained in Ω

z-Transform

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC (μ is the unit step sequence)
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left|\alpha z^{-1}\right| < 1$$

- ROC is the annular region $|z| > |\alpha|$

z-Transform

- Example - The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left|\alpha z^{-1}\right| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } \left|z^{-1}\right| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

z-Transform

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

z-Transform

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region $|z| < |\alpha|$

z-Transform

- Note: The z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

Table 3.8: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational z-Transforms

- The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational z-Transforms

- A rational z-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the ***zeros*** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the ***poles*** of $G(z)$

Rational *z*-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M}(z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N}(z - \lambda_\ell)}$$

- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

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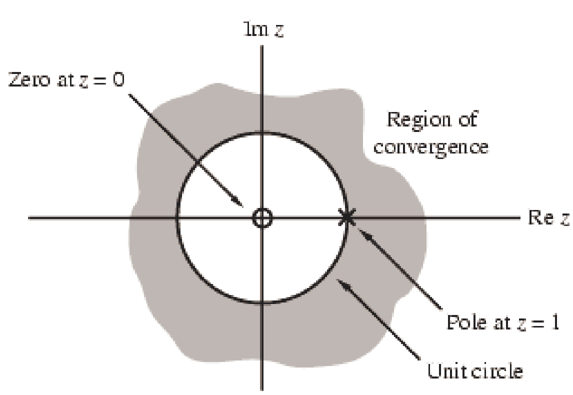
Rational *z*-Transforms

- <u>Example</u> - The z-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$



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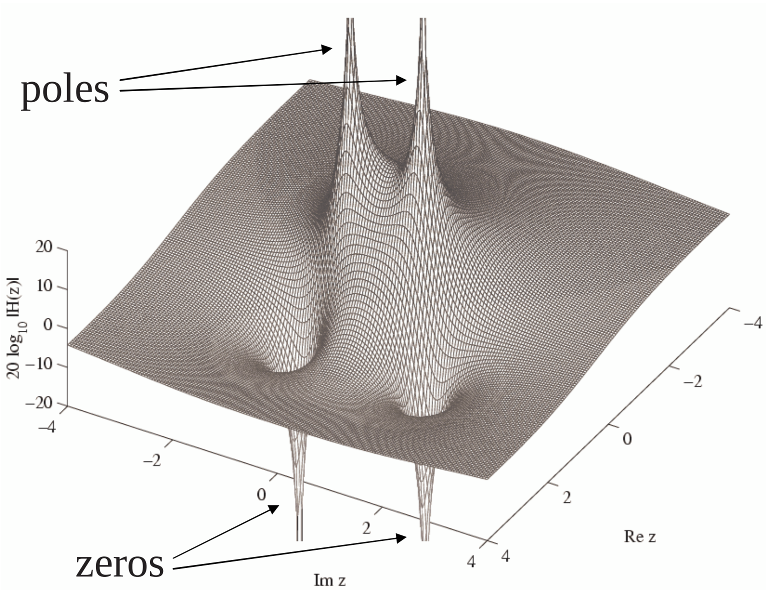
Rational *z*-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

Rational *z*-Transforms



Rational z-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

ROC of a Rational z-Transform

- ROC of a z-transform is an important concept
- <u>Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform</u>
- Hence, the z-transform must always be specified with its ROC

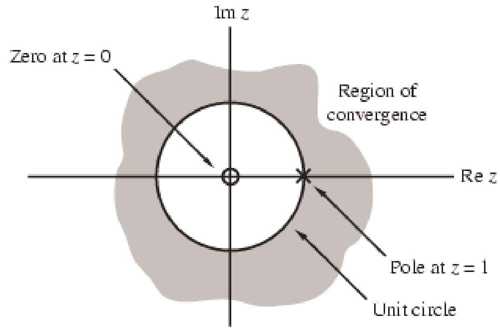
ROC of a Rational z-Transform

- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$

ROC of a Rational *z*-Transform

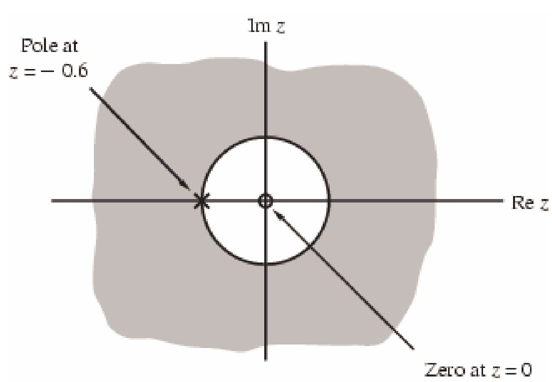


- In this plot, the ROC, shown as the shaded area, is the region of the *z*-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational *z*-Transform

- <u>Example</u> - The *z*-transform $H(z)$ of the sequence is given by $h[n] = (-0.6)^n \mu[n]$

$$H(z) = \frac{1}{1 + 0.6\, z^{-1}}, \quad |z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational z-Transform

- A sequence can be one of the following types: ***finite-length, right-sided, left-sided and two-sided***
- In general, the ROC depends on the type of the sequence of interest

ROC of a Rational z-Transform

- Example - Consider a ***finite-length sequence*** $g[n]$ defined for $-M \le n \le N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n]\, z^{-n} = \frac{\sum_{n=0}^{N+M} g[n-M]\, z^{N+M-n}}{z^{N}}$$

ROC of a Rational z-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$
- As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$

ROC of a Rational z-Transform

- Example - A ***right-sided sequence*** with nonzero sample values for $n \geq 0$ is sometimes called a ***causal sequence***
- Consider a causal sequence $u_1[n]$
- Its z-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

ROC of a Rational z-Transform

- Example - A ***left-sided sequence*** with nonzero sample values for $n \leq 0$ is sometimes called a ***anti-causal sequence***
- Consider an anti-causal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \le N$ with N nonnegative has a z-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

ROC of a Rational z-Transform

- Example - The z-transform of a ***two-sided sequence*** $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

ROC of a Rational *z*-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n]z^{-n}$, can be interpreted as the *z*-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the *z*-transform does not exist

ROC of a Rational *z*-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex
- Its *z*-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

ROC of a Rational z-Transform

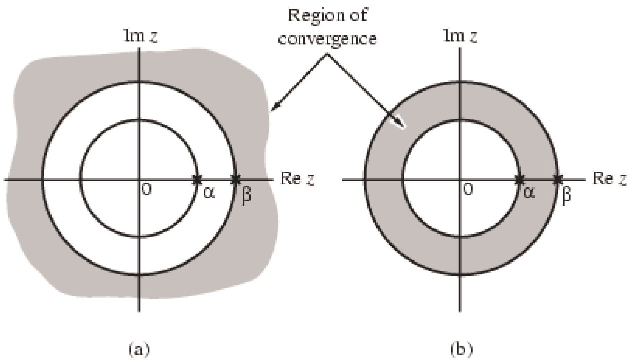
- There is no overlap between these two regions
- Hence, the z-transform of $u[n] = \alpha^n$ does not exist!

ROC of a Rational z-Transform

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
- As an example, assume that a rational z-transform $X(z)$ has two simple poles at $z = \alpha$ and $z = \beta$ with $|\alpha| < |\beta|$
- There are three possible ROCs associated with $X(z)$

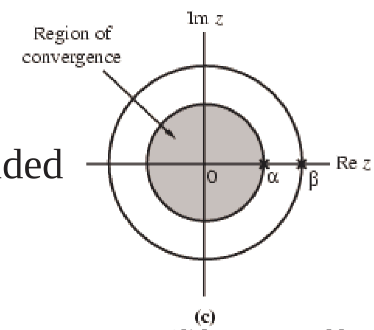
ROC of a Rational *z*-Transform

Right-sided



Two-sided

Left-sided



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ROC of a Rational *z*-Transform

- The ROC of a rational *z*-transform can be easily determined using MATLAB

  `[z,p,k] = tf2zp(num,den)`

  determines the zeros, poles, and the constant of a rational *z*-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

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ROC of a Rational z-Transform

[num,den] = zp2tf(z,p,k) implements the reverse process

- The factored form of the *z*-transform can be obtained using sos = zp2sos(z,p,k)

ROC of a Rational z-Transform

- The ***pole-zero plot*** is determined using the function zplane
- The *z*-transform can be either described in terms of its zeros and poles: zplane(zeros,poles)

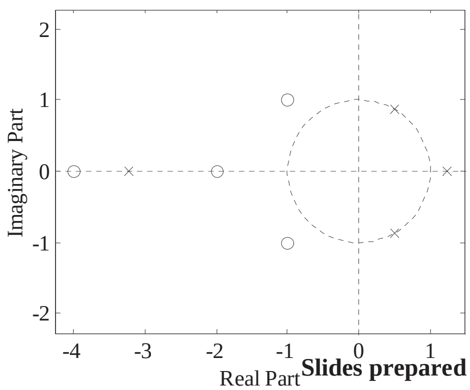
  or, it can be described in terms of its numerator and denominator coefficients: zplane(num,den)

ROC of a Rational z-Transform

- Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below



× − pole
o − zero

Inverse z-Transform

- ***General Expression***: Recall that, for $z = r e^{j\omega}$, the z-transform $G(z)$ given by

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g[n] r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g[n] r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{j\omega})\, e^{j\omega n} d\omega$$

Inverse z-Transform

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

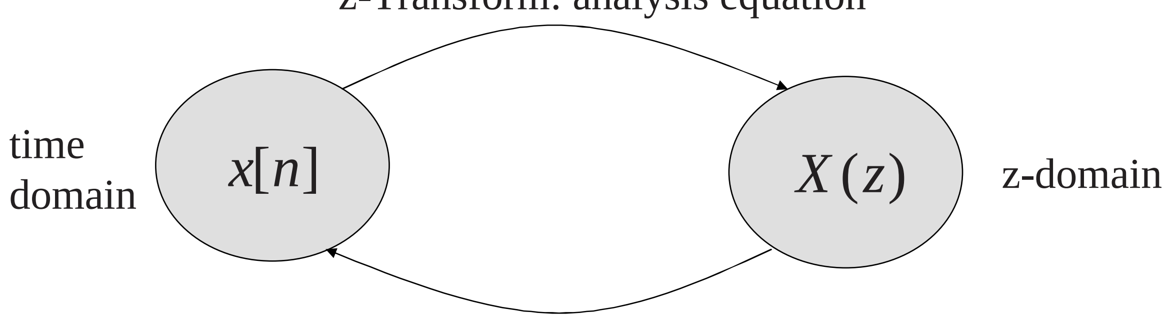
$$g[n] = \frac{1}{2\pi j}\oint_{C'} G(z)\, z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

z-Transform: analysis equation



Inverse z-Transform: synthesis equation

$$x[n] = \frac{1}{2\pi j}\oint_{C'} X(z)\, z^{n-1} dz$$

Inverse z-Transform

- But the integral remains unchanged when it is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the ***Cauchy's residue theorem*** resulting in

$$g[n] = \sum_{z_i \ inside\ C} \mathrm{Res}(G(z)z^{n-1})_{|z=z_i}$$

- The above equation needs to be evaluated at all values of n and is not pursued here

Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has a ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a ***proper fraction***
- <u>Example</u> - Consider

$$G(z) = \frac{2 + 0.8 z^{-1} + 0.5 z^{-2} + 0.3 z^{-3}}{1 + 0.8 z^{-1} + 0.2 z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5 z^{-1} + \frac{5.5 + 2.1 z^{-1}}{1 + 0.8 z^{-1} + 0.2 z^{-2}}$$

Inverse Transform by Partial-Fraction Expansion

- ***Simple Poles:*** In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of G(z) be at $z = \lambda_k, 1 \leq k \leq N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

Inverse Transform by Partial-Fraction Expansion

- The constants ρ_ℓ in the partial-fraction expansion are called the ***residues*** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) G(z) \big|_{z=\lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has a ROC given by $|z| > |\lambda_\ell|$ and, thus, has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Note: The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence

Inverse Transform by Partial-Fraction Expansion

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

Inverse Transform by Partial-Fraction Expansion

- Now

$$\rho_1 = (1 - 0.2\,z^{-1})H(z)\Big|_{z=0.2} = \left.\frac{1 + 2\,z^{-1}}{1 + 0.6\,z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1 + 0.6\,z^{-1})H(z)\Big|_{z=-0.6} = \left.\frac{1 + 2\,z^{-1}}{1 - 0.2\,z^{-1}}\right|_{z=-0.6} = -1.75$$

Inverse Transform by Partial-Fraction Expansion

- Hence

$$H(z) = \frac{2.75}{1 - 0.2\,z^{-1}} - \frac{1.75}{1 + 0.6\,z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- ***Multiple Poles***: If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, $1 \le \ell \le N - L$

Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z)\right]_{z=\nu},$$

$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Partial-Fraction Expansion Using MATLAB

[r,p,k]= residuez(num,den) develops the partial-fraction expansion of a rational *z*-transform with numerator and denominator coefficients given by vectors num and den

- Vector r contains the residues
- Vector p contains the poles
- Vector k contains the constants η_ℓ

Partial-Fraction Expansion Using MATLAB

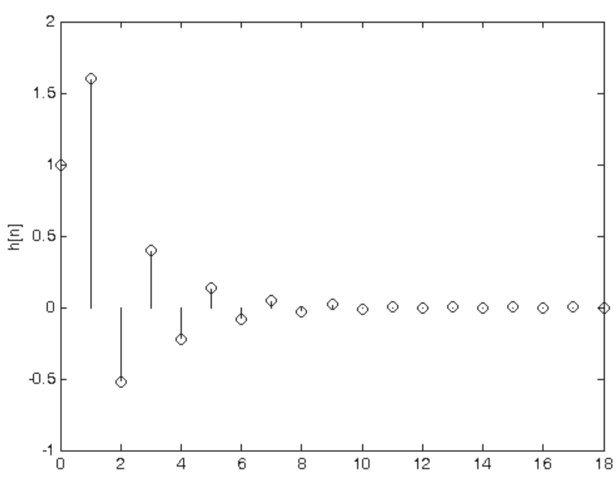
[num,den]=residuez(r,p,k) converts a *z*-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform Using MATLAB

- The function impz can be used to find the inverse of a rational *z*-transform *G*(*z*)
- The function computes the coefficients of the power series expansion of *G*(*z*)
- The number of coefficients can either be user specified or determined automatically

Inverse z-Transform Using MATLAB

```
>> num=[1 2];
>> den=[1 0.4 -0.12];
>> [h,t]=impz(num,den);
>> figure(1)
>> stem(t,h)
>> xlabel('n')
>> ylabel('h[n]')
```



$h[n]$=[1.0000 1.6000 -0.5200 0.4000 -0.2224 …]

Table 3.9: z-Transform Properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert \mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z \dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j} \oint_C G(v) H(z/v) v^{-1}\, dv$	Includes $\mathcal{R}_g \mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j} \oint_C G(v) H^*(1/v^*) v^{-1}\, dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g \mathcal{R}_h$ denotes the region $R_{g^-} R_{h^-} < |z| < R_{g^+} R_{h^+}$.