Computational Diagnostics & Biocybernetics Unit

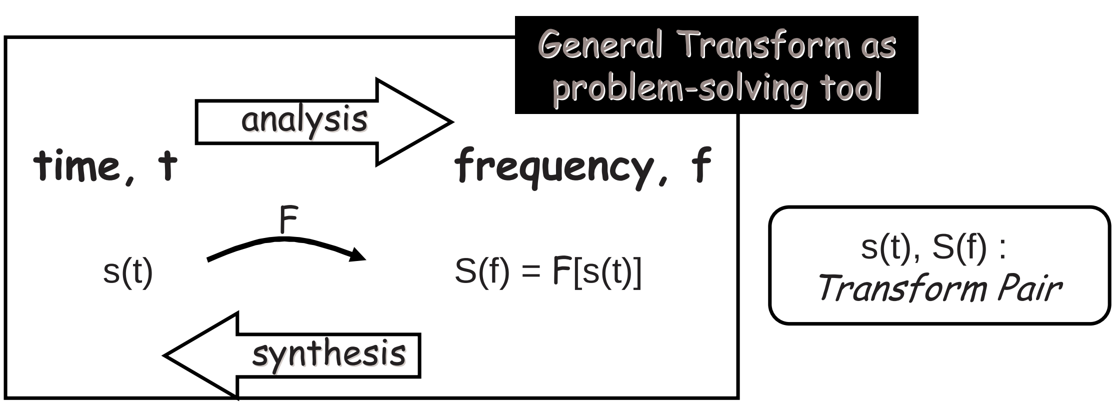
BMT 403 :
Einführung in die Biosignalverarbeitung
Ergänzende Folien 2

Prof. Dr. Dr. Daniel J. Strauss

based on slides from M. E. Angoletta, CERN

Frequency analysis: why?

- Fast & efficient insight on signal's building blocks.
- Simplifies original problem - ex.: solving Part. Diff. Eqns. (PDE).
- Powerful & complementary to time domain analysis techniques.
- Several transforms in DSPing: Fourier, Laplace, z, etc.

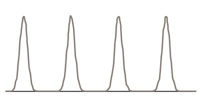


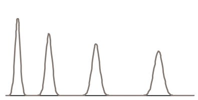
Fourier analysis - applications

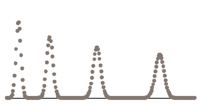
Applications wide ranging and ever present in modern life

- *Telecomms* - GSM/cellular phones,
- *Electronics/IT* - most DSP-based applications,
- *Entertainment* - music, audio, multimedia,
- *Accelerator control* (tune measurement for beam steering/control),
- *Imaging, image processing,*
- *Industry/research* - X-ray spectrometry, chemical analysis (FT spectrometry), PDE solution, radar design,
- *Medical* - (PET scanner, CAT scans & MRI interpretation for sleep disorder & heart malfunction diagnosis,
- *Speech analysis* (voice activated “devices”, biometry, …).

Fourier analysis - tools







Input Time Signal			Frequency spectrum	
Continuous	Periodic ***(period T)***	FS	Discrete	$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$
	Aperiodic	FT	Continuous	$S(f)=\int_{-\infty}^{+\infty} s(t)\cdot e^{-j2\pi f t}dt$
Discrete	Periodic ***(period T)***	DFS**	Discrete	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$
	Aperiodic	DTFT	Continuous	$S(f)=\sum_{n=-\infty}^{+\infty} s[n]\cdot e^{-j2\pi f n}$
		DFT**	Discrete	$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$

Note: $j=\sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, ***N = No. of samples***

** ***Calculated via FFT***

A little history

- Astronomic predictions by Babylonians/Egyptians likely via trigonometric sums.

- **1669**: Newton stumbles upon light spectra (*specter* = ghost) but fails to recognise "frequency" concept (*corpuscular* theory of light, & no waves).

- **18th century**: two outstanding problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation). **BUT** peers' consensus is that sum of sinusoids *only* represents smooth curves. Big blow to utility of such sums for all but Fourier ...

- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines. Strong criticism by peers blocks publication. Work published, 1822 *("Theorie Analytique de la chaleur"*).

A little history -2

- **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

CONTINUOUS

- → Fourier extends the analysis to arbitrary function (Fourier Transform).
- → Dirichlet, Poisson, Riemann, Lebesgue address FS convergence.
- → Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

DISCRETE: Fast calculation methods (FFT)

- → **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).
- → **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).
- → Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).
- → FFT algorithm refined & modified for most computer platforms.

Fourier Series (FS)

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

synthesis

$$s(t) = a_0 + \sum_{k=1}^{+\infty} \left[a_k \cdot \cos(k\omega t) - b_k \cdot \sin(k\omega t) \right]$$

For all t but discontinuities

a_0, a_k, b_k : Fourier coefficients.

k: harmonic number,

T: period, $\omega = 2\pi/T$

analysis

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)\,dt$$

(signal average over a period, i.e. DC term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\omega t)\,dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\omega t)\,dt$$

***** see next slide**

FS convergence

Dirichlet conditions

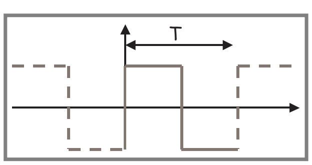
In any period:

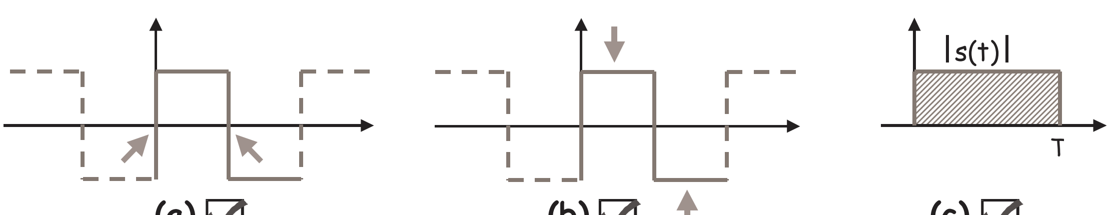
(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

(c) s(t) absolutely integrable , $\int_0^T |s(t)|\,dt < \infty$

Example:
square wave





(a) ☑ **(b)** ☑ **(c)** ☑

$x \le y$, then $f(x) \le f(y)$

FS analysis - 1

FS of odd* function: square wave.

$T = 2\pi \quad \Rightarrow \omega = 1$

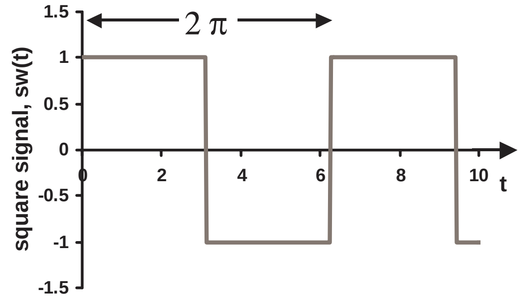
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\cos kt\,dt - \int_{\pi}^{2\pi}\cos kt\,dt\right\} = 0$$ *(odd function)*

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi}\sin kt\,dt - \int_{\pi}^{2\pi}\sin kt\,dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot\{1-\cos k\pi\} =$$

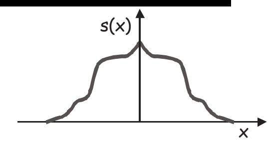
$$= \begin{cases} \dfrac{4}{k\cdot\pi}, & k\ \text{odd} \\ 0, & k\ \text{even} \end{cases}$$

$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$

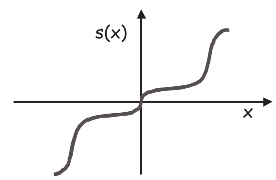


*** Even & Odd functions**

Even : $s(-x) = s(x)$



Odd : $s(-x) = -s(x)$



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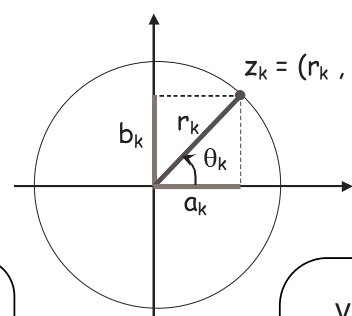
FS analysis - 2

Fourier spectrum representations

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$

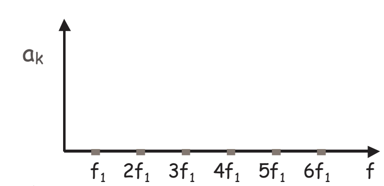


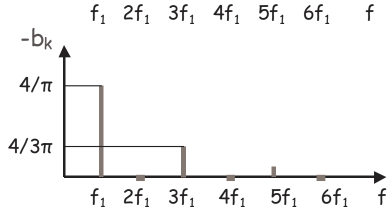
$r_k = \sqrt{a_k^2 + b_k^2}$

$\theta_k = \arctan(b_k/a_k)$

Rectangular

$v_k = \mathbf{a_k}\cos(\omega k\, t) - \mathbf{b_k}\sin(\omega k\, t)$





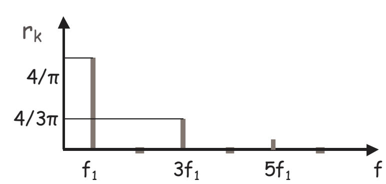
$\mathbf{r_K}$ = amplitude,
θ_K = phase

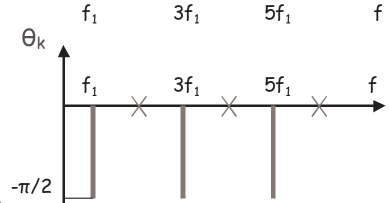
$f_k = k\,\omega/2\pi$

Fourier spectrum of square-wave.

Polar

$v_k = \mathbf{r_k}\cos(\omega k\, t + \theta_k)$

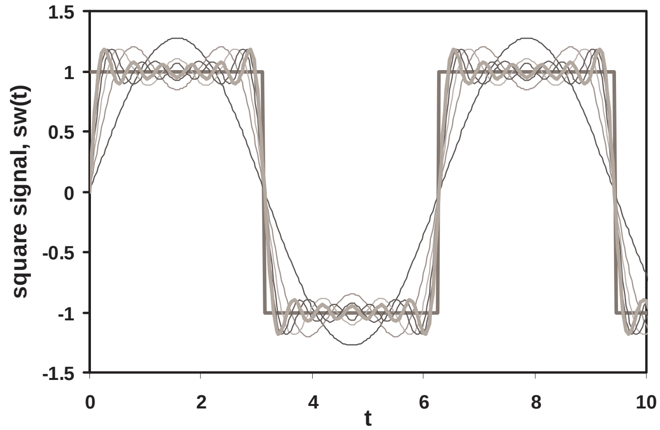




FS synthesis

Square wave reconstruction from spectral terms

$$sw_{11}(t)=\sum_{k=1}^{11}\left[-b_k \cdot \sin(kt)\right]$$

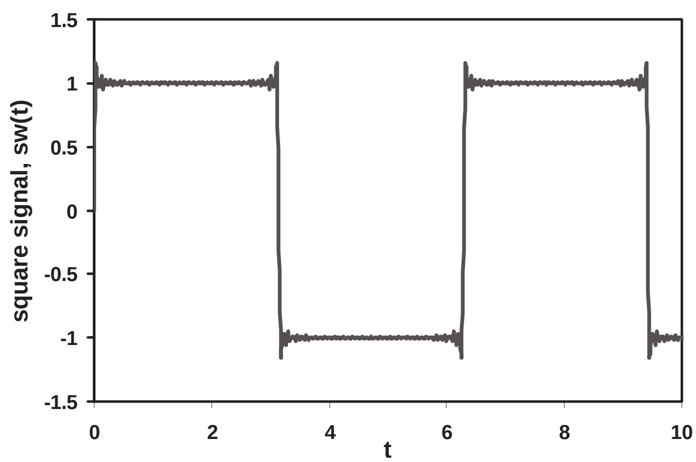


Convergence may be slow (~1/k) - ideally need infinite terms.
Practically, series truncated when remainder below computer tolerance (⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

Gibbs phenomenon

Overshoot exist @ each discontinuity

$$sw_{79}(t)=\sum_{k=1}^{79}\left[-b_k \cdot \sin(kt)\right]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude.
  Just a minor annoyance.

FS time shifting

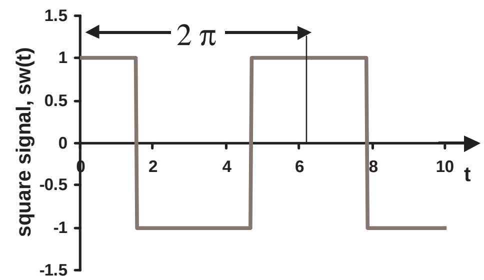
FS of even function: $\pi/2$-advanced square-wave

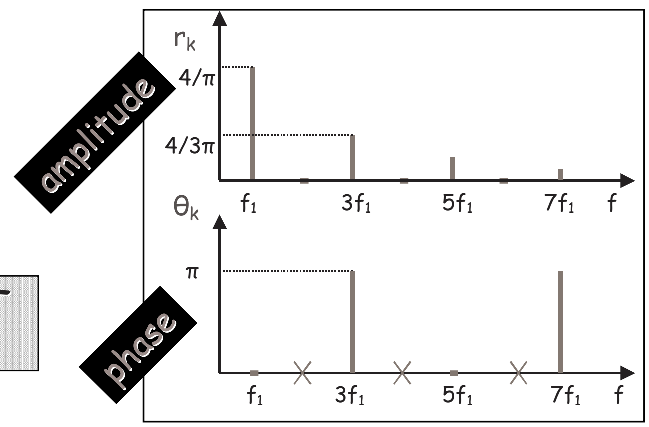
$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \dfrac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 1, 5, 9\ldots \\ -\dfrac{4}{k\cdot\pi}, & k \text{ odd}, \quad k = 3, 7, 11\ldots \\ 0, & k \text{ even.} \end{cases}$$

$-b_k = 0$ *(even function)*

Note: amplitudes unchanged ***BUT*** phases advance by $k\cdot\pi/2$.





Complex FS

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - j\cdot\sin(t)$ ⟹ "phasor" $\quad \cos(t) = \dfrac{e^{jt} + e^{-jt}}{2} \quad \sin(t) = \dfrac{e^{jt} - e^{-jt}}{2\cdot j}$

analysis

$$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-jk\omega t}dt$$

synthesis

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

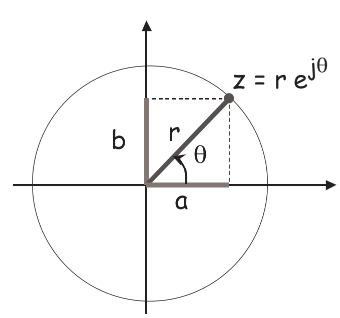
Complex form of FS (Laplace 1782). Harmonics c_k separated by $\Delta f = 1/T$ on frequency plot.

Note: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

$c_0 = a_0$

$$c_k = \frac{1}{2}\cdot(a_k + j\cdot b_k) = \frac{1}{2}\cdot(a_{-k} - j\cdot b_{-k})$$



$r = \sqrt{a^2 + b^2}$

$\theta = \arctan(b/a)$

FS properties

	Time	Frequency
Homogeneity	$a \cdot s(t)$	$a \cdot S(k)$
Additivity	$s(t) + u(t)$	$S(k)+U(k)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(k)+b \cdot U(k)$
Time reversal	$s(-t)$	$S(-k)$
Multiplication *	$s(t) \cdot u(t)$	$\sum_{m=-\infty}^{\infty} S(k-m)U(m)$
Convolution *	$\frac{1}{T} \cdot \int_0^T s(t-\bar{t}) \cdot u(\bar{t})\, d\bar{t}$	$S(k) \cdot U(k)$
Time shifting	$s(t-\bar{t})$	$e^{-j\frac{2\pi k \cdot \bar{t}}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi m t}{T}} \cdot s(t)$	$S(k - m)$

FS - power

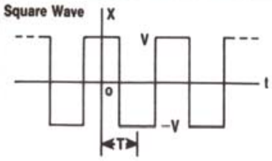
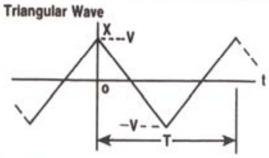
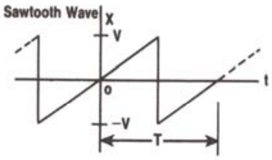
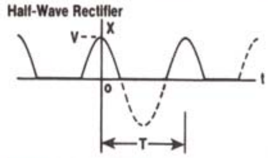
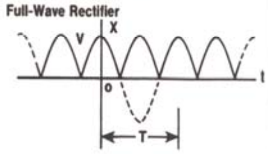
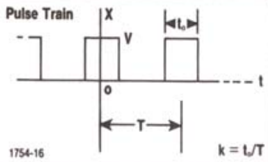
Average power W : $W = \frac{1}{T} \int_o^T |s(t)|^2 \, dt \equiv s(t) \otimes s(t)$

Parseval's Theorem

$$W = \sum_{k=-\infty}^{\infty} |c_k|^2 = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2 \right)$$

- FS convergence ~1/k
  - ⇒ *lower frequency terms*
    $W_k = |c_k|^2$ carry most power.
- W_k vs. ω_k: Power density spectrum.

FS of main waveforms

Wave Shape	Fourier Series -- $\omega_o = 2\pi/T$
Square Wave 	$x(t)=\frac{4V}{\pi}(\cos\omega_o t-\frac{1}{3}\cos 3\omega_o t+\frac{1}{5}\cos 5\omega_o t-\frac{1}{7}\cos 7\omega_o t+\ldots)$
Triangular Wave 	$x(t)=\frac{8V}{\pi^2}(\cos\omega_o t+\frac{1}{9}\cos 3\omega_o t+\frac{1}{25}\cos 5\omega_o t+\ldots)$
Sawtooth Wave 	$x(t)=\frac{2V}{\pi}(\sin\omega_o t-\frac{1}{2}\sin 2\omega_o t+\frac{1}{3}\sin 3\omega_o t-\frac{1}{4}\sin 4\omega_o t+\ldots)$
Half-Wave Rectifier 	$x(t)=\frac{V}{\pi}(1+\frac{\pi}{2}\cos\omega_o t+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\ldots\ldots(-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\ldots)$ n even
Full-Wave Rectifier 	$x(t)=\frac{2V}{\pi}(1+\frac{2}{3}\cos 2\omega_o t-\frac{2}{15}\cos 4\omega_o t+\frac{2}{35}\cos 6\omega_o t-\ldots\ldots(-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_o t\ldots)$ n even
Pulse Train  1754-16 $k = t_o/T$	$x(t)=V[k+\frac{2}{\pi}(\sin k\pi\cos\omega_o t+\frac{1}{2}\sin 2k\pi\cos 2\omega_o t+\ldots\ldots+\frac{1}{n}\sin nk\pi\cos n\omega_o t+\ldots)]$ $k = t_o/T$

Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

DFS defined as:

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ same period N

i.e. time periodicity propagates to frequencies!

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi k n}{N}}$$

Synthesis: finite sum ⇐ band-limited s[n]

DFS generate periodic c_k with same signal period

Orthogonality in DFS

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

↑ Kronecker's delta

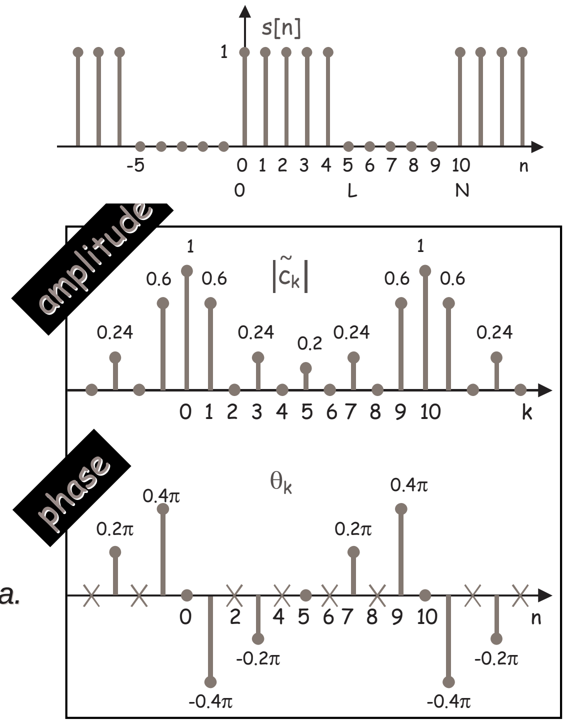
N consecutive samples of s[n] completely describe s in time or frequency domains.

DFS analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\[2ex] \dfrac{e^{-j\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$



Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function

DFS properties

	Time	Frequency
Homogeneity	$a \cdot s[n]$	$a \cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication *	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution *	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-j\frac{2\pi k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi h t}{T}} \cdot s[n]$	$S(k - h)$

** Explained in next week's lecture*

TOPICS

1. **Infinite Fourier Transform (FT)**
2. **FT & generalised impulse**
3. **Uncertainty principle**
4. **Discrete Time Fourier Transform (DTFT)**
5. **Discrete Fourier Transform (DFT)**
6. **Comparing signal by DFS, DTFT & DFT**

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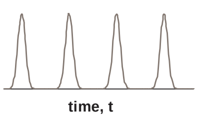
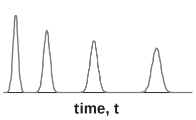
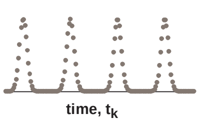
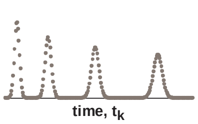
Fourier analysis - tools









Input Time Signal			Frequency spectrum	
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Continuous	Aperiodic	FT	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$
Discrete	Periodic *(period T)*	DFS	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$
Discrete	Aperiodic	DTFT	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$
Discrete	Aperiodic	DFT	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$

***Note:** $j = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n] = s(t_n)$, N = No. of samples*

Fourier Integral (FI)

Fourier analysis tools for aperiodic signals.

Fourier Integral Theorem

Any aperiodic signal s(t) can be expressed as a Fourier integral if s(t) piecewise smooth[(1)] in any finite interval (-L,L) and absolute integrable[(2)].

$$s(t) = \int_0^{+\infty} \{A(\omega)\cdot\cos(\omega t) + B(\omega)\cdot\sin(\omega t)\} d\omega \quad \textbf{(3)}$$

(1) s(t) continuous, s'(t) monotonic

(2) $\int_{-\infty}^{+\infty} |s(t)|\, dt < +\infty$

(3) $A(\omega) = \frac{1}{\pi}\cdot\int_{-\infty}^{+\infty} s(t)\cos(\omega t)\, dt \qquad B(\omega) = \frac{1}{\pi}\cdot\int_{-\infty}^{+\infty} s(t)\sin(\omega t)\, dt$

Real-to-complex link

$$S(\omega) = \pi\cdot[A(\omega) - j\cdot B(\omega)]$$

Complex form

analysis

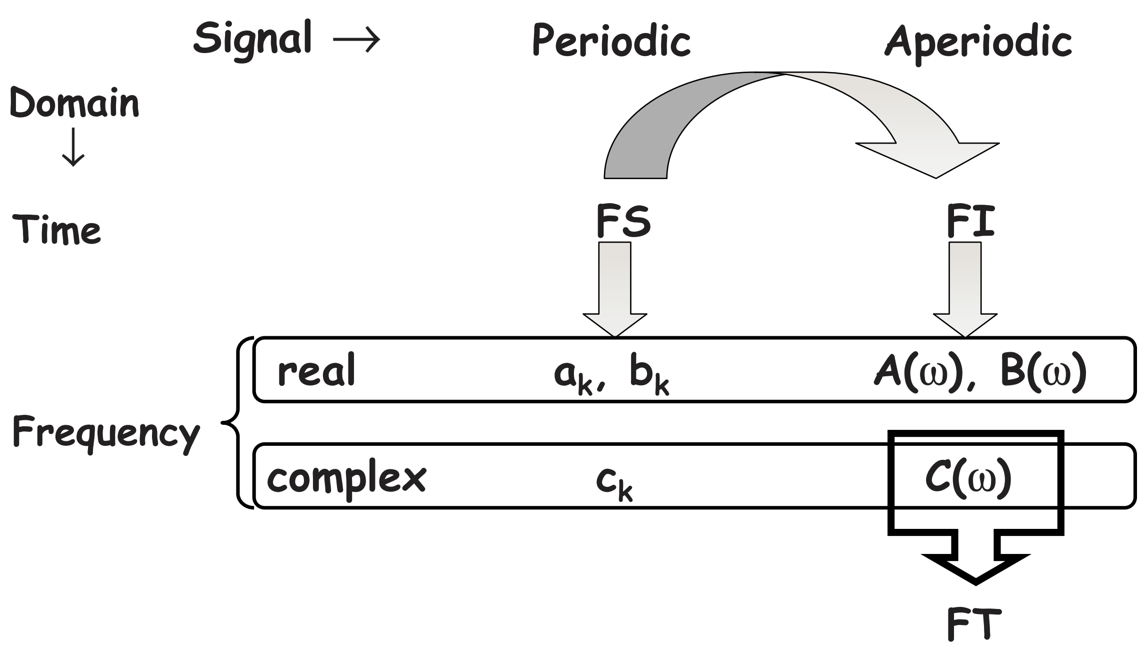
$$S(\omega) = \int_{-\infty}^{+\infty} s(t)\cdot e^{-j\omega t}\, dt$$

synthesis

$$s(t) = \frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} S(\omega)\cdot e^{j\omega t}\, d\omega$$

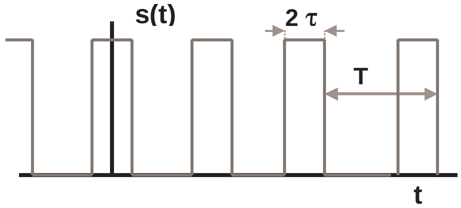
Fourier Transform (Pair) - FT

Let's summarise a little



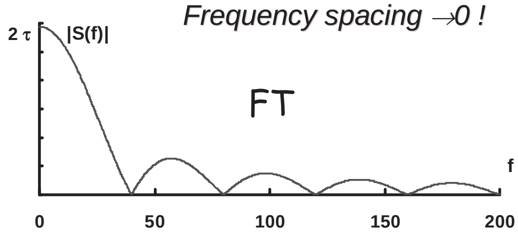
From FS to FT

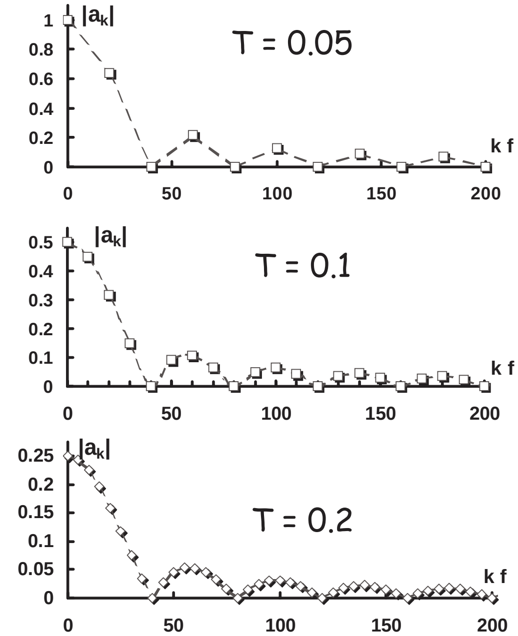
FS moves to FT as period T increases: ➡ continuous spectrum



Pulse train, width $2\tau = 0.025$

Frequency spacing →0 !





Note: $|a_k| \to 2\,a_0$ as $k \to 0 \Rightarrow 2\,a_0$ is plotted at k=0

Computational Diagnostics & Biocybernetics Unit

Getting FT from FS

FS defined

$$c_k = \frac{1}{T} \cdot \int_{-T/2}^{T/2} s(t) \cdot e^{-jk\omega t}\, dt$$

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega t}$$

$\Delta f = \Delta\omega/(2\pi) = 1/T$
frequency spacing

1

$$c_k = \Delta f \cdot \int_{-T/2}^{T/2} s(t) \cdot e^{-j\,\omega_k\, t} dt \equiv c_{\omega_k}$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} c_{\omega_k} \cdot e^{j\,\omega_k\, t}$$

2

$$\Gamma_{\omega_k} \equiv \frac{c_{\omega_k}}{\Delta f} = \int_{-T/2}^{T/2} s(t) \cdot e^{-j\,\omega_k\, t} dt$$

$$s(t) = \sum_{\omega_k=-\infty}^{\infty} \Gamma_{\omega_k} \cdot e^{j\,\omega_k\, t}\, \Delta f$$

As $\Delta f \to 0$, replace Δf, ω_K, $\sum_{\omega_k=-\infty}^{\infty}$

by df, $2\pi f$, $\int_{-\infty}^{+\infty}$

FT defined

$$S(f) = \lim_{\Delta f \to 0} \Gamma_{\omega_k} = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t}\, dt$$

$$s(t) = \int_{-\infty}^{+\infty} S(f) \cdot e^{j2\pi\, ft}\, df$$

FT properties

	Time	Frequency
Linearity	$a\cdot s(t) + b\cdot u(t)$	$a\cdot S(f)+b\cdot U(f)$
Multiplication	$s(t)\cdot u(t)$	$\int_{-\infty}^{+\infty} S(f-\bar{f})\cdot U(\bar{f})\,d\bar{f}$
Convolution	$\int_{-\infty}^{+\infty} s(t-\bar{t})\cdot u(\bar{t})\,d\bar{t}$	$S(f)\cdot U(f)$
Time shifting	$s(t-\bar{t})$	$e^{-j2\pi f\bar{t}}\cdot S(f)$
Frequency shifting	$e^{+j2\pi\bar{f}}\cdot s(t)$	$S(f-\bar{f})$
Time reversal	$s(-t)$	$S(-f)$
Differentiation	$\frac{ds(t)}{dt}$	$j2\pi f\, S(f)$
Parseval's identity	$\int h(t)\, g^*(t)\, dt = \int H(f)\, G^*(f)\, df$	
Integration	$\int_{-\infty}^{t} s(u)\,du$	$S(f)/(j2\pi f)$
Energy & Parseval's (E is t-to-f invariant)	$E = \int_{-\infty}^{+\infty} \lvert s(t)\rvert^2 dt = \int_{-\infty}^{+\infty} \lvert S(f)\rvert^2 df$	

FT - uncertainty principle

Bandwidth Theorem

For effective duration Δt & bandwidth Δf

$\exists\, \gamma > 0 \quad \Delta t\cdot\Delta f \geq \gamma$

uncertainty product

Fourier uncertainty principle

For Energy Signals:

$E=\int |s(t)|^2 dt = \int |S(f)|^2 df < \infty$

Define mean values

$$\bar{t}^{\,2} = \frac{1}{E}\cdot \int_{-\infty}^{+\infty} t\cdot |s(t)|^2\, dt \qquad \bar{f}^{\,2} = \frac{1}{E}\cdot \int_{-\infty}^{+\infty} f\cdot |S(f)|^2\, df$$

Define std. dev.

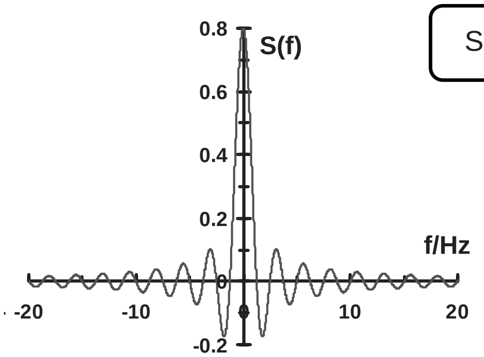
$$\Delta t^2 = \frac{1}{E}\cdot \int_{-\infty}^{+\infty} (t-\bar{t})\cdot |s(t)|^2\, dt \qquad \Delta f^2 = \frac{1}{E}\cdot \int_{-\infty}^{+\infty} (f-\bar{f})\cdot |S(f)|^2\, df$$

$\Rightarrow$ **$\Delta t\cdot\Delta f \geq 1/4\pi$**

Implications

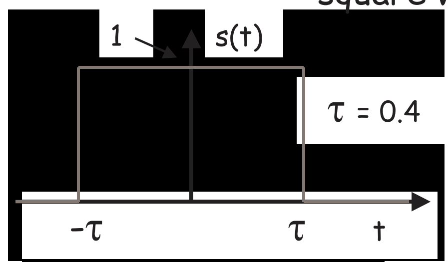
- Limited accuracy on simultaneous observation of s(t) & S(f).
- Good time resolution (small Δt) requires large bandwidth Δf & vice-versa.

FT - example



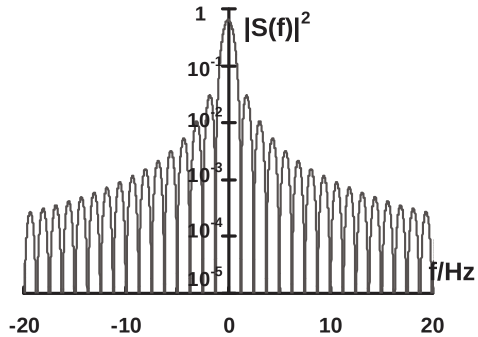
S(f) = 2τ s_{MAX} sync(2fτ)

FT of 2τ-wide square window



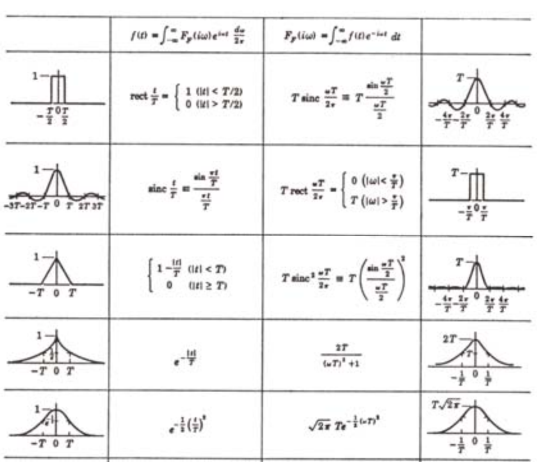
Fourier uncertainty

Choose
Δt = |∫s(t)/s(0) dt| = 2τ,
Δf = |∫S(f)/S(0) df|=1/(2τ) = half distance btwn first 2 zeroes ($f_{1,-1}$ = ±1/2τ) of S(f)
then: **Δt · Δf = 1**

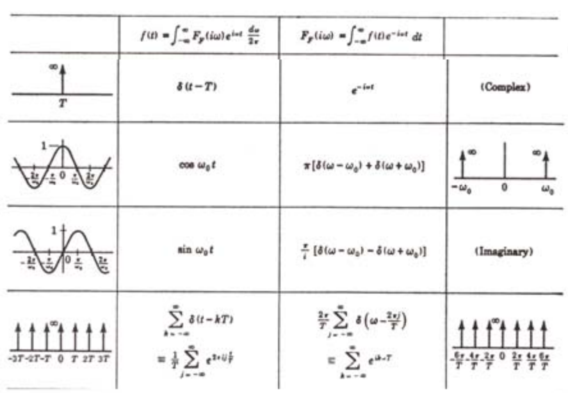


Power Spectral Density (PSD) vs. frequency f plot.
Note: Phases unimportant!

FT of main waveforms



	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$	
	$\text{rect}\,\frac{t}{T} = \begin{cases} 1 & (\lvert t\rvert < T/2) \\ 0 & (\lvert t\rvert > T/2) \end{cases}$	$T \,\text{sinc}\, \frac{\omega T}{2\pi} \equiv T \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\text{sinc}\, \frac{t}{T} \equiv \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T \,\text{rect}\, \frac{\omega T}{2\pi} = \begin{cases} 0 & (\lvert\omega\rvert < \frac{\pi}{T}) \\ T & (\lvert\omega\rvert > \frac{\pi}{T}) \end{cases}$	
	$\begin{cases} 1 - \frac{\lvert t\rvert}{T} & (\lvert t\rvert < T) \\ 0 & (\lvert t\rvert \geq T) \end{cases}$	$T \,\text{sinc}^2 \frac{\omega T}{2\pi} \equiv T \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$	
	$e^{-\frac{\lvert t\rvert}{T}}$	$\frac{2T}{(\omega T)^2 + 1}$	
	$e^{-\frac{1}{2}\left(\frac{t}{T}\right)^2}$	$\sqrt{2\pi}\, T e^{-\frac{1}{2}(\omega T)^2}$	



	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$	
	$\delta(t - T)$	$e^{-i\omega t}$	(Complex)
	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	
	$\sin \omega_0 t$	$\frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	(Imaginary)
	$\sum_{k=-\infty}^{\infty} \delta(t - kT) \equiv \frac{1}{T} \sum_{j=-\infty}^{\infty} e^{2\pi i j \frac{t}{T}}$	$\frac{2\pi}{T} \sum_{j=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi j}{T}\right) \equiv \sum_{k=-\infty}^{\infty} e^{ik\omega T}$	

Discrete Time FT (DTFT)

DTFT defined as:

Note: continuous frequency domain!
(frequency density function)

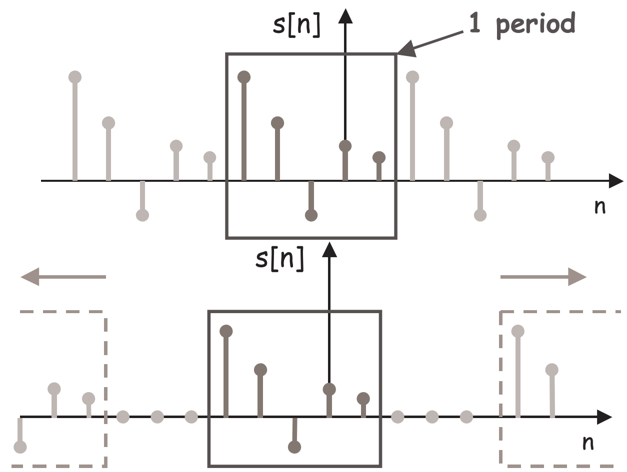
analysis

$$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$$

synthesis

$$s[n] = \frac{1}{2\pi} \cdot \int_{0}^{2\pi} S(f) e^{j2\pi f n} df$$

Holds for aperiodic signals

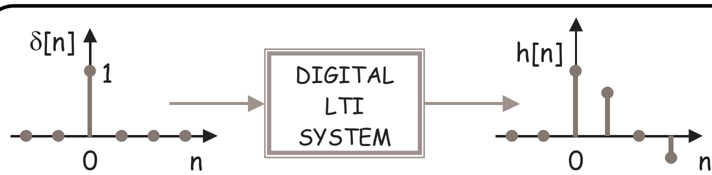


Obtained from DFS as N → ∞

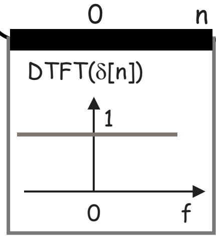
$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi k n}{N}}$$

DTFT - convolution

Digital Linear Time Invariant system: obeys superposition principle.



h[t] = impulse response



Convolution

$$y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[n-m] \cdot h[m]$$

x[n]	h[n]	y[n]
⇕	⇕	⇕
X(f)	H(f)	Y(f) = X(f) · H(f)

kernel of FT as eigenfunction of LTI systems – see blackboard

Discrete FT (DFT)

DFT defined as:

analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-j\frac{2\pi kn}{N}}$$

synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{j\frac{2\pi kn}{N}}$$

Frequency resolution

Analysis frequencies f_m

$$f_m = \frac{m\cdot f_S}{N}, m = 0, 2 \ldots N-1$$

- Applies to discrete time and frequency signals.
- Same form of DFS but for aperiodic signals:
  signal treated as periodic for computational purpose only.

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ spectrum has period N

DFT properties

	Time	**Frequency**
Linearity	$a\cdot s[n] + b\cdot u[n]$	$a\cdot S(k)+b\cdot U(k)$
Multiplication	$s[n]\cdot u[n]$	$\frac{1}{N}\cdot\sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution	$\sum_{m=0}^{N-1} s[m]\cdot u[n-m]$	$S(k)\cdot U(k)$
Time shifting	$s[n-m]$	$e^{-j\frac{2\pi k\cdot m}{T}}\cdot S(k)$
Frequency shifting	$e^{+j\frac{2\pi ht}{T}}\cdot s[n]$	$S(k-h)$

⋮